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A NOTE ON MEROMORPHIC FUNCTIONS[1]

BY

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1. A relation satisfied by every polynomial $f(z)$ is[2]

$$\lim_{r\to\infty} \frac{\log M(r)}{n(r)\log r} = 1 \qquad \dots (1)$$

This relation does not hold[3] for integral functions even if we consider functions of zero order only. I prove below a number of results from which we deduce (i) If $f(z)$ be an integral function of zero order we have

$$\liminf_{r\to\infty} \frac{\log M(r)}{n(r, a)\log r} \leqslant 1 \qquad \dots (2)$$

for every a (ii) If $f(z)$ be an integral function of zero order and if $n(r, f) \sim A(l_2 r)^{\alpha_2} \cdots (l_k r)^{\alpha_k}$ where $A > 0$, $\alpha_2 > 0$, then[4] (1) holds.

2. For rational functions we have either

$$\lim_{r\to\infty} \frac{T(r)}{n(r, 0)\log r} = 1 \qquad \dots (3)$$

or

$$\lim_{r\to\infty} \frac{T(r)}{n(r, \infty)\log r} = 1 \qquad \dots (4)$$

According as $n(r, 0) >$ or $\leqslant n(r, \infty)$. We prove below corresponding results for meromorphic functions.

3. THEOREM 1. If $f(z)$ be a meromorphic function of order ρ, where $0 \leqslant \rho < 1$ then

$$\liminf_{r\to\infty} \frac{T(r)}{N(r, a) + N(r, b)} \leqslant \frac{1}{1-\rho} \qquad \dots (5)$$

where $a \neq b$ are any two numbers[5] $(0 \leqslant |a| \leqslant \infty)$.

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1. An abstract of this note appeared in *Proc. Indian Sc. Congress* (1943) Calcutta.
2. For the notations $M(r)$ etc. see E. C. Titchmarsh *Theory of Functions* (1939) Ch 8.
3. For analagous results see S. M. Shah (i) *London Math. Soc. J.* 15 (1940) 25-31 (ii) *American Math. Soc. Bulletin* 46 (1940) 909-913 (iii) *Indian Math. Soc. J.* 5 (1941) 179-188 (iv) *ibid* 5, 189-191 (v) *Proc. National Acad. of Sciences, Allahabad* 12 (1942) 9-12.
4. If $\alpha_2 = \ \ = \alpha_{s-1} = 0$ then $\alpha_s > 0$; $l_2 r = \log\log r$ and so on.
5. For a similar result, when $\rho > 1$ is non-integral see R. Nevanlinna *Fonctions Meromorphes.* p. 51-55.

THEOREM 2. If $f(z)$ be an integral function of order ρ, $0 \leqslant \rho < 1$ then

$$\liminf_{r\to\infty} \frac{\log M(r)}{N(r, a)} \leqslant \frac{1}{1-\rho} \qquad \ldots (6)$$

where a is any finite number.

Cor. If $f(z)$ be of zero order then

$$\liminf_{r\to\infty} \frac{\log M(r)}{N(r, a)} = 1 \qquad \ldots (7)$$

$$\liminf_{r\to\infty} \frac{\log M(r)}{n(r, a)\log r} \leqslant 1 \qquad \ldots (8)$$

for every finite a.

4. For a polynomial $f(z)$ we have

$$\lim_{r\to\infty} \frac{\log M(r)}{\{n(r, a)+n(r, b)\}\log r} = \frac{1}{2} \qquad \ldots (9)$$

where a and b are any two finite numbers. For integral functions of finite order we have a corresponding result[1]

$$\limsup_{r\to\infty} \frac{\log M(r/k)}{\{n(r, a)+n(r, b)\}\log r} < \infty \qquad \ldots (10)$$

where $k > 1$ is a constant $a \neq b$

However we cannot replace in (10), $\log r$ by $\log r/\theta(r)$ where $\theta(r)$ is *any* increasing function tending to infinity with r. In fact we have.

THEOREM 3. Given $\rho \geqslant 0$, there exists an Integral function of order ρ for which

$$\limsup_{r\to\infty} \frac{\log M(r/k)}{\{n(r, a)+n(r, b)\}\log r} \geqslant \frac{1}{2} \qquad \ldots (11)$$

where $a \neq b$ are any two finite numbers.

5. If $n(r)$ increases fairly regularly we have

THEOREM 4. If $f(z)$ be an integral function of zero order and if $n(r) \sim A(l_1 r)^{\alpha_1} \cdots (l_k r)^{\alpha_k}$

then

$$\lim_{r\to\infty} \frac{\log M(r)}{n(r)\log r} = \frac{1}{1+\alpha_1} \qquad \ldots (12)$$

THEOREM 5. If $f(z)$ be a meromorphic function of zero order and if

$$n(r, 0) \sim A(l_1 r)^{\alpha_1} \cdots (l_k r)^{\alpha_k};\ n(r, \infty) = o(n(r, 0))$$

then

$$\lim_{r\to\infty} \frac{T(r)}{n(r, 0)\log r} = \frac{1}{1+\alpha_1} \qquad \ldots (13)$$

1. G. Valiron: *Lectures on the general theory of Integral Functions* p. 81. For Meromorphic Functions R. Nevanlinna *loc. cit.* p. 69.

If $n(r, \infty) \sim A(l_k r)^{\alpha_1} \cdots (l_k r)^{\alpha_k}$; $n(r, 0) = o(n(r, \infty))$

then
$$\lim_{r\to\infty} \frac{T(r)}{n(r, \infty)\log r} = \frac{1}{1+\alpha_1} \qquad \ldots (14)$$

6. Proof of Theorem 1. We may suppose without loss of generality $a=0$, $b=\infty$. Let

$$f(z) = A\, z^p \frac{\Pi(1 - z/a_i)}{\Pi(1 - z/b_j)} = A\, z^p \frac{f_1(z)}{f_2(z)}$$

where a_i and b_j are zeros and poles of $f(z)$ other than those at origin. We suppose $n(x, f_1) + n(x, f_2)$ tends to infinity with x. For $r > r_0$

$$T(r) < A \log r + \log M(r, f_1) + \log M(r, f_2)$$

$$\leqslant \left\{1 + o(1)\right\}\left[\int_0^r \frac{n(x, f_1) + n(x, f_2)}{x}\, dx + r\int_r^\infty \frac{n(x, f_1) + n(x, f_2)}{x^2}\, dx\right]$$

$$= \left\{1 + o(1)\right\}\left\{I(r) + r\, J(r)\right\} \text{ say}$$

$$= \left\{1 + o(1)\right\}\left\{r\int_r^\infty \frac{I(x)}{x^2}\, dx\right\}$$

Now $\lim\limits_{x\to\infty} \dfrac{I(x)}{x^{\rho+\varepsilon}} = 0$ for every $\varepsilon > 0$. Hence we can choose a sequence of values $R_n \to \infty$ such that

$$\frac{I(x)}{x^{\rho+\varepsilon}} \leqslant \frac{I(R_n)}{R_n^{\rho+\varepsilon}} \quad \text{for } x > R_n.$$

Let $R_n = R$. Then $T(R) \leqslant \{(1 + o(1)\} \dfrac{I(R)}{1-(\rho+\varepsilon)}$

Hence $\liminf\limits_{r\to\infty} \dfrac{T(r)}{I(r)} \leqslant \dfrac{1}{1-\rho}$ which proves (5).

The proof of Theorem 2 is similar and the corollary follows from (6) and Jensen's Inequality.

7. Proof of Theorem 3. We first consider $\rho > 0$. Let

$$\lambda_1 = 1;\ P = 1 + [\rho];\ \lambda_{n+1} = e^{\lambda_n^P};\ f(z) = \sum \left(\frac{z}{\lambda_n}\right)^{\left[\lambda_n^\rho\right]}$$

$f(z)$ is an integral function of order ρ. Further since $\left(\dfrac{z}{\lambda_n}\right)^{\left[\lambda_n^\rho\right]}$ is the maximum term of the series $f(z)$ when $z = \lambda_{n+1}$, we have by Rouche's Theorem for $n \geqslant n_0$, $n(\lambda_{n+1}, a) + n(\lambda_{n+1}, b) = 2\left[\lambda_n^\rho\right]$. Further

$$\log M\left(\frac{\lambda_{n+1}}{k}\right) \sim \log\left(\frac{\lambda_{n+1}}{k\,\lambda_n}\right)^{\left[\lambda_n^\rho\right]} \sim \lambda_n^\rho\left[\lambda_n^P - \log k - \log \lambda_n\right] \sim \lambda_n^{\rho+P}.$$

Hence for $r=\lambda_{n+1}$

$$\frac{\log M(r|k)}{\{n(r,a)+n(r,b)\}\log r} \sim \frac{\lambda_n^{\rho+P}}{2\lambda_n^{\rho+P}} = \frac{1}{2}$$

which proves the theorem when $\rho>0$. If $\rho=0$ we take

$$f(z)=\sum\left(\frac{z}{\lambda_n}\right)^{[e^{\sqrt{(\log \lambda_n)}}]}$$

The proofs of Theorems (4) and (5) are straightforward and are omitted.

ON DIFFERENTIAL EQUATIONS REDUCIBLE TO CLAIRAUT'S FORM

BY

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1. Many differential equations may be reduced to Clairaut's form $Y=PX+F(P)$, where X, Y are functions of x, y and $P\equiv\frac{dY}{dX}$, and the solution immediately written down viz: $Y=cX+F(c)$. In this paper are discussed equations reducible to Clairaut's form by the substitutions $Y=y^s$, $X=x^t$.

2. Consider the equation

$$Y-PX=f(P)/\phi(P) \qquad \ldots \quad (1)$$

where

$$f(P)=\alpha_0 P^n-\alpha_1 P^{n-1}+\cdots\cdots(-1)^n\alpha_n;$$

$$\phi(P)=a_1 P^{n-1}-a_2 P^{n-2}+\cdots\cdots(-1)^{n+1}a_n$$

are polynomials in P with constant coefficients.

By the substitutions $Y=y^s$, $X=x^t$, $P=\frac{sy^{s-1}}{tx^{t-1}}p$ the equation becomes after simplification

$$\left(\frac{sy^{s-1}}{tx^{t-1}}\right)^n(a_1x^t+\alpha_0)p^n+\sum_{r=1}^{n-1}(-1)^r\left(\frac{sy^{s-1}}{tx^{t-1}}\right)^{n-r}(a_r y^s+a_{r+1}x^t+\alpha_r)p^{n-r}$$
$$+(-1)^n(a_n y^s+\alpha_n)=0 \qquad \ldots \quad (2)$$

which is of the nth degree in p, and the solution is

$$y^s-c\,x^t=f(c)/\phi(c) \qquad \ldots \quad (3)$$

3. A. *To investigate when an equation of the form*

$$R_0\, p^n + R_1\, p^{n-1} + \cdots R_n = 0$$

can be reduced to the form (1) *by the substitutions* $Y = y^s$, $X = x^t$.

By comparing with 2 the conditions to be satisfied are:—

(*a*) R_n must be of the form $x^{l'} y^{m'} (a_0 x^{t'} + \alpha_0)$, where a_0 or α_0 may be zero; and R_n of the form $x^{l''} y^{m''} (a_n y^{s'} + \alpha_n)$, where a_n or α_n may be zero, R_r is the sum of 3 or less terms of the type $ax^l y^m$, for $r = 1, 2 \cdots (n-1)$.

(*b*) $R_0 \ldots R_n$ must satisfy the identity

$$R_0 + k\frac{x}{y} R_1 + k^2 \frac{x^2}{y^2} R_2 + \cdots + k^n \frac{x^n}{y^n} R_n \equiv$$

$$x^l y^m \left[\beta_0 + \beta_1 \frac{x^t}{y^s} + \beta_2 \frac{x^{2t}}{y^{2s}} + \cdots + \beta_n \frac{x^{nt}}{y^{ns}}\right] \quad \ldots \quad (4)$$

where $k = s/t$ and l, m, $\beta_0 \cdots \beta_n$ are any constants (including zero).

In regard to (4) it must be noted that:—

(i) Each term on the right should emerge as a remainder from the corresponding term on the left,

i.e. $(x^l y^m \beta_0)$ from R_0; $(x^l y^m \cdot \beta_1 x^t/y^s)$ from $kR_1\, x/y$ etc.

(ii) No term on the right is of the same degree in x or y.

(iii) The condition is satisfied if the right-hand side is a single term.

4. *To solve the equation when the above conditions are satisfied.*

We may find the values of s, t by one or more of the following methods:—

(*a*) Suppose R_r can be put in the form $x^l y^m (a_r y^{s'} + a_{r+1} x^{t'} + \alpha_r)$

Then if a_r, a_{r+1} α_r are all not zero, then $s = s'$, $t = t'$ (no other values are possible).

If $\alpha_r = 0$, but a_r, $a_{r+1} \neq 0$, then $s = \pm s'$ and $t = \pm t'$;

If $a_{r+1} = 0$, but a_r, $\alpha_r \neq 0$ then $s = \pm s'$;

If $a_r = 0$ but a_{r+1}, $\alpha_r \neq 0$ then $t = \pm t'$.

(*b*) If R_n consists of a single term, let the equation be arranged so that R_0 contains no factor in y. If then y^i is a factor of R_n, we must have

$$s = \frac{n-i}{n-1} \text{ or } s = \frac{n-i}{n} \quad \ldots \quad (5)$$

Similarly, if R_0 consists of a single term, let the equation be arranged so that R_n contains no factor in x. If then x^j is a factor of R_0, we have

$$t = \frac{n-j}{n-1} \text{ or } t = \frac{n-j}{n} \qquad \cdots \quad (6)$$

We thus obtain at most two values for each of s, t. The correct values may be determined from (4).

Having found the values of s, t, the equation may then be easily put in the form (2) and the solution written down by means of (3)

* **Examples:** (i) $y^2 p^3 + 2xp - y = 0.$

$$R_0 + k\frac{x}{y}R_1 + k^2\frac{x^2}{y^2}R_2 + k^3\frac{x^3}{y^3}R_3 \equiv y^2 + 2k^2\frac{x^3}{y^2} - k^3\frac{x^3}{y^2}.$$

Condition (4) is satisfied if $2k^2 - k^3 = 0$, *i.e.* $k = 2$.

Dividing throughout by y^2, we see that y^{-1} is a factor of R_3.

$\therefore$ from (5) $s = \frac{3+1}{3-1}$ or $\frac{3+1}{3}$ *i.e.* $s = 2$ or $\frac{4}{3}$

Similarly from (6), $t = 3/2$ or 1. Since $k = s/t = 2$ we have $s = 2$, $t = 1$.

The equation may be written

$$(2y)^3(0x+1)p^3 - (2y)^2(0y^2 + 0x + 0)p^2 + (2y)(0y^2 + 8x + 0)p - (8y^2 + 0) = 0$$

The solution is $y^2 - cx = \frac{c^3}{8}$.

(ii) $p^3 - pxy^4 - y^5 = 0$

As above, we obtain $k = -1$, $s = -1$ or $-\frac{2}{3}$, $t = \frac{3}{2}$, or 1. $\therefore s = -1$, $t = 1$.

Solution: $y^{-1} - cx = \frac{c^3}{-1}$

(iii) $axy\,p^2 + (x^2 - ay^2 - b)\,p - xy = 0$

From R_1 we see that $s = 2$, $t = 2$., $k = 1$.

$$R_0 + k\frac{x}{y}R_1 + k^2\frac{x^2}{y^2}R_2 = -bxy.$$

$\therefore$ Condition (4) is satisfied. Solution: $y^2 - cx^2 = \frac{-bc}{ac+1}$

* Taken from Forsyth's "*Treatise on Differential Equations*" Pages 34 and 50.

ON RECIPROCAL SCREWS

BY

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1. It is well known that the null lines of a system of forces in three dimensions constitute a linear complex. In this paper, reciprocal screws are regarded from this standpoint.

Two screws are said to be reciprocal if a wrench acting on either does no work when the body is twisted about the other. The analytical condition is $(p+p')\cos\theta+h\sin\theta=0$, where p,p' are the pitches, and θ, h denote respectively the angle and shortest distance between the screws.

Let us *define* reciprocal screws then as follows:

Two screws are reciprocal if their associated linear complexes are conjugate. ... (1·1)

Two linear complexes Ω and Ω' given by the equations

$$\Omega:\ ap_{12}+bp_{13}+cp_{14}+dp_{34}+ep_{42}+fp_{23}=0$$

$$\Omega:\ a'p_{12}+b'p_{13}+c'p_{14}+d'p_{34}+e'p_{42}+f'p_{23}=0$$

where the p denote line coordinates, are *conjugate* if

$$ad'+a'd+be'+b'e+cf'+c'f=0$$

The special property of such conjugate linear complexes is that

If l is a line belonging to the linear complex r and l' the polar of l with respect to another linear complex r', then l' also belongs to r if and only if r, r' are conjugate. ... (1·2)

Suppose now that Ω, Ω' are the systems of null lines of two given wrenches. Then, if Ω, Ω' are conjugate, we can find lines AB, CD which are null lines for the first and polar for the second. The second wrench is therefore equivalent to two forces say, P, P′, acting along AB and CD respectively.

Let OX be the axis of the first wrench, and p its pitch. Consider a small displacement consisting of a rotation $\delta\phi$ about OX and a translation $p\,\delta\phi$ parallel to OX. Let d be the shortest distance

between OX and AB and θ the angle between them. The work done by the force P along AB in this displacement is

$$P \sin \theta \cdot \delta\phi + P \cos \theta \cdot p \ \delta\phi = \frac{P\delta\phi}{R}(R \sin \theta \cdot d + pR \cos \theta)$$

where R is the force of any wrench on the screw OX.

But $R \sin \theta \cdot d + pR \cos \theta =$ moment of the first system about AB, and is equal to zero, since AB is a null line of that system. Similarly, the work done by the force P′ along CD is zero. Thus, our definition of reciprocal screws, is equivalent to the usual one given in books on mechanics.

2. The method used above leads to two interesting results:

I. Two screws are reciprocal if and only if wrenches on them may be replaced by conjugate forces acting along two pairs of opposite edges of the same tetrahedron.

Let Ω, Ω' be associated with wrenches on reciprocal screws. Let AB, CD be two lines of Ω which are also polar w. r. t. Ω'. Then if AC is any transversal of these lines, its polar w. r. t. Ω must also be a transversal, say BD. Hence the first wrench may be replaced by forces along AC, BD and the second by forces along AB, CD.

Conversely, consider two pairs of forces acting along AB, CD and AC, BD respectively. Then taking ABCD as tetrahedron of reference, the linear complexes associated with each pair have equations of the form:

$$ap_{12} + dp_{34} = 0 \text{ and } b' p_{13} + e' p_{42} = 0.$$

which are obviously conjugate, and therefore the corresponding screws must be reciprocal.

II. Three screws are mutually reciprocal if and only if wrenches on them may be replaced by conjugate forces acting along the three pairs of opposite edges of the same tetrahedron.

For if Ω'' is reciprocal to both Ω and Ω', the polar line of AD which is a null line of Ω and Ω', must also be a null line of both, and hence must be BC.

ON SOME IDENTITIES AND A GENERALIZATION OF WILSON'S THEOREM

BY

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1. THEOREM. Let $a_0, a_1, \cdots$ be an arithmetic progression (A.P.) with common difference d. Let $\phi(\theta)=(x+a_0)(x+a_1)\cdots(x+a_\theta)$; and $\psi(\theta)=d\cdot 2d, \ldots \theta d$, x being arbitrary; then

$$\frac{{}_nC_r}{x+a_0}-\frac{{}_nC_{r+1}}{x+a_1}+\cdots+(-1)^{n-r}\frac{{}_nC_n}{x+a_{n-r}}=$$

$$\frac{\psi(n-r)}{\phi(n-r)}+{}_rC_1\frac{\psi(n-r-1)}{\phi(n-r-1)}+\cdots+{}_{n-1}C_{n-r}\frac{\psi(0)}{\phi(0)} \quad \ldots \quad (1)$$

Proof: We prove the theorem by induction. Assuming it true when $n=m$, $r=\lambda$, and when $n=m$, $r=\lambda-1$, we write down the two corresponding results; and observing that ${}_nC_r+{}_nC_{r-1}={}_{n+1}C_r$, we see that the theorem is true when $n=m+1$, $r=\lambda$. Also by resolving $\psi(n)/\phi(n)$ into partial fractions, since $a_0, a_1, \cdots$ are in A. P. with common difference d, we see that

$$\frac{{}_nC_0}{x+a_0}-\frac{{}_nC_1}{x+a_1}+\cdots+(-1)^n\cdot\frac{{}_nC_n}{x+a_n}=\frac{\psi(n)}{\phi(n)}, \quad \ldots \quad (2)$$

so that the theorem is true for all n when $r=0$.

Again since $\dfrac{{}_nC_n}{x+a_0}=\dfrac{\psi(0)}{\phi(0)}$, it is also true when $r=n$.

The above three results are easily seen to be sufficient to establish the truth of the theorem completely.

2. We next derive some interesting identities from the above.

(i) Substituting for $\dfrac{\psi(n-r)}{\phi(n-r)}$, and similar expressions on the right side of (1), their values as given by (2), we get

$$\frac{{}_nC_r}{x+a_0}-\frac{{}_nC_{r+1}}{x+a_1}+\cdots+(-1)^{n-r}\cdot\frac{{}_nC_n}{x+a_{n-r}}=\left[\sum_{\theta=0}^{n-r}\frac{(-1)^\theta\cdot{}_{n-r}C_\theta}{x+a_\theta}\right]$$

$$+{}_rC_1\left[\sum_{\theta=0}^{n-r-1}(-1)^\theta\cdot\frac{{}_{n-r-1}C_\theta}{x+a_\theta}\right]+\cdots+{}_{n-2}C_{n-r-1}\left[\sum_{\theta=0}^{1}(-1)^\theta\cdot\frac{{}_1C_\theta}{x+a_\theta}\right]+{}_{n-1}C_{n-r}$$

Multiplying both sides of this identity by $x+a_{\lambda-r}$, and putting $x=-a_{\lambda-r}$ we get

$${}_nC_\lambda={}_{n-r}C_{\lambda-r}+{}_rC_1\cdot{}_{n-r-1}C_{\lambda-r}+\cdots+{}_{n-\lambda+r-1}C_{n-\lambda}\cdot{}_{\lambda-r}C_{\lambda-r}.$$

(ii) Again put $x=0$ $a_0=1$ $a_1=2, \cdots$ in (1). Then giving r successively the values 1, 2, . . . n, we get

$$ {}_nC_1 - \frac{1}{2}\,{}_nC_2 + \cdots + (-1)^{n-1}.\frac{{}_nC_n}{n} = \frac{1}{n} + \frac{1}{n-1}\,{}_1C_1 + \cdots + {}_{n-1}C_{n-1} \qquad [A] $$

$$ {}_nC_2 - \frac{1}{2}\cdot{}_nC_3 + \cdots + (-1)^{n-2}.\frac{{}_nC_n}{n-1} = \frac{1}{n-1} + \frac{1}{n-2}\cdot{}_2C_1 + \cdots + {}_{n-1}C_{n-2} \qquad [B] $$

. . . .

$$ {}_nC_n = 1 \qquad [K] $$

Thus $[A]-[B]+[C]- \cdots +(-1)^{n-1}\cdot[K]$ gives

$$ {}_nC_1 - {}_nC_2\left(1+\frac{1}{2}\right) + {}_nC_3\left(1+\frac{1}{2}+\frac{1}{3}\right) - \cdots + (-1)^{n-1}\cdot{}_nC_n\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right) $$

$$ = \frac{1}{n} + ({}_1C_1 - 1)\cdot\frac{1}{n-1} + ({}_2C_2 - {}_2C_1 + 1)\frac{1}{n-2} + \cdots + [{}_{n-1}C_{n-1} - \cdots + (-1)^{n-1}] $$

$=1/n$, the other terms vanishing.

This is an interesting identity. We may similarly get other relations. For example, $[A]-\frac{1}{2}\cdot[B]+ \cdots +(-1)^{n-1}\cdot\frac{1}{n}$ [K] gives

$$ {}_nC_1 - {}_nC_2\left[\frac{1}{1\cdot2}+\frac{1}{2\cdot1}\right] + {}_nC_3\left[\frac{1}{1\cdot3}+\frac{1}{2\cdot2}+\frac{1}{3\cdot1}\right] - \cdots + (-1)^{n-1}\left[\frac{1}{1\cdot n}+\frac{1}{2\cdot n-1}+\cdot\right] $$

$$ = \frac{1}{n} + \frac{1}{2\cdot n-1} + \frac{1}{3\cdot n-2} + \cdots + \frac{1}{n-1\cdot2} + \frac{1}{n}\,. $$

(iii) Putting in (1) $r=0$, $a_0=0$, $a_1=1$, . . . we have

$$ \frac{{}_nC_0}{x} - \frac{{}_nC_1}{x+1} + \frac{{}_nC_2}{x+2} + \cdots + (-1)^n\cdot\frac{{}_nC_n}{x+n} = \frac{n!}{x(x+1)\cdots(x+n)}. $$

For $|x|<1$, this gives when x changed to $-x$

$$ -\frac{{}_nC_0}{x} - {}_nC_1(1+x+x^2+x^3+\cdots) + \frac{1}{2}\cdot{}_nC_2\left(1+\frac{x}{2}+\frac{x^2}{4}+\cdots\right) $$

$$ + \cdots + (-1)^n\frac{1}{n}\left[1+\frac{x}{n}+\frac{x^2}{n^2}+\cdots\right] $$

$= -\frac{1}{x}\left[1+S_1^n\cdot x+S_2^n x^2+\cdots\right]$, where S_r^n denotes the sum of the products of the reciprocal of 1, 2, . . . n taken r at a time.

Equating coefficients of like powers of x, we get

$$ {}_nC_1 - \frac{{}_nC_2}{2^k} + \frac{{}_nC_3}{3^k} - \cdots + (-1)^{n-1}.\frac{{}_nC_n}{n^k} = S_k^n\,;\ (k=1, 2, \cdots) \quad \ldots \quad (3) $$

$$ \text{Hence } S_1^n - \frac{1}{2}S_2^n + \cdots = {}_nC_1\cdot\left[1-\frac{1}{2}+\frac{1}{3}-\cdot\right] - {}_nC_2\left[\frac{1}{2}-\frac{1}{2^2\cdot2}+\frac{1}{2^3\cdot3}-\cdot\right] $$

$$ + \cdots + (-1)^{n-1}\left[\frac{1}{n}-\frac{1}{n^2\cdot2}+\cdots\right] $$

$$ = {}_nC_1\cdot\log 2 - {}_nC_2\log\left(1+\frac{1}{2}\right) + \cdots + (-1)^{n-1}\cdot\log\left(1+\frac{1}{n}\right) $$

$$ = {}_{n+1}C_2\cdot\log 2 - {}_{n+1}C_3\log 3 + \cdots + (-1)^{n-1}\cdot\log(n+1) $$

(iv) Using the relation (3) we can show that for $k=1, 2, \ldots$

$$[(n-1)!]^{2k}\cdot\left[\overset{n}{S}_{2k}-\frac{1}{n^{2k}}\right]\equiv 0 \text{ mod } (n^2), \text{ where } n \text{ is an odd prime.}$$

For. the left side, which is evidently an integer,

$$=(n-1)!^{2k}\left(\frac{{}_nC_1}{1^{2k}}-\frac{{}_nC_2}{2^{2k}}+\cdots+-\frac{{}_nC_{n-1}}{(n-1)^{2k}}\right)$$

$$=[(n-1)!]^{2k}\left(\frac{{}_nC_1}{(1\cdot n-1)^{2k}}[(n-1)^{2k}-1]-\cdots\right.$$

$$\left.+\frac{(-1)^{(n+1)/2}\cdot{}_nC_{(n-1)/2}}{\left[\frac{n-1}{2}\cdot n-\frac{n-1}{2}\right]^{2k}}\left[\left(n-\frac{n-1}{2}\right)^{2k}-\left(\frac{n-1}{2}\right)^{2k}\right]\right)$$

Since n is prime ${}_nC_1, \ldots$ are all divisible by n, as also the numbers $(n-1)^{2k}-1$, $(n-2)^{2k}-2^{2k}, \ldots\ldots$ Hence the result.

3. A generalisation of Wilson's Theorem: †

If n is prime and $\lambda+r\equiv 0$ mod n, then for $r\leqslant n$,

$$\frac{\lambda(\lambda+1)\cdots(\lambda+n)}{\lambda+r}\cdot\frac{{}_nC_r}{n}+(-1)^r\equiv 0 \text{ mod } n. \tag{A}$$

Proof. We have $\dfrac{\lambda(\lambda+1\cdots(\lambda+n)}{\lambda+r}$

$$=[(\overline{\lambda+r}-r)\cdot(\overline{\lambda+r}-\overline{r-1})\cdots(\overline{\lambda+r}-1)][(\overline{\lambda+r}+1)\cdots(\overline{\lambda+r}+n-r)]$$

$$=(-1)^r\cdot r!\,(n-r)! \text{ mod } n \text{ since } \lambda+r\equiv 0 \text{ mod } n.$$

Also $\dfrac{{}_nC_r}{n}=\dfrac{n!}{(n-r)!\,r!}\cdot\dfrac{1}{n}=\dfrac{(n-1)!}{(n-r)!\,r!}$.

Hence the left side of (A) $\equiv(-1)^r\cdot\left[r!\,(n-r)!\,\dfrac{(n-1)!}{(n-r)!\,r!}+1\right]$ mod n.

$$=(-1)^r[(n-1)!+1]\equiv 0 \text{ mod } n.$$

† The Theorem of §3 was obtained from §1 and §2. The elegant proof given here is due to Dr. R. Vaidyanathaswamy.

GLEANINGS

$(a+b)^2$

Marriage is the square of *a* plus *b*
In other words
*a*² plus *b*² plus 2*ab*
Where 2*ab* (of course)
Are twins

Poem by Christopher Morley.

AN EXTENSION OF A THEOREM OF STEINER

BY

P. KESAVA MENON

The object of this note is to give an extension of the following theorem of Steiner:

If two triangles ABC, $A'B'C'$ be such that the perpendiculars from A, B, C to $B'C'$, $C'A'$, $A'B'$ respectively are concurrent B', then the perpendiculars from A', B', C' to BC, CA, AB respectively are also concurrent.

The proposed extension is as follows:

If two sets of three circles a, b, c; a', b', c' be such that the three circles coaxal with b, c; c, a; a, b and orthogonal to a'; b'; c' respectively are coaxal, then the three circles coaxal with b', c'; c', a'; a', b' and orthogonal to a; b; c respectively are also coaxal.

The proof is based on the representation of circles by points of space of three dimensions. The result in three dimensions is easily seen to be as follows:

If a, b, c; a', b', c' be two triads of points in space and the conjugate points with respect to a quadric S of a', b', c' lying on bc, ca, ab respectively are collinear then the conjugate points with respect to S of a, b, c lying on $b'c'$, $c'a'$, $a'b'$ respectively are also collinear.

We shall adopt the following notations:

$P_a=0$ is the equation to the polar plane of a with respect to S.

(ab) is the value of P_a when the coordinates of b are substituted in it.

Then the ratios in which the planes $P_{a'}$, $P_{b'}$, $P_{c'}$ divide bc, ca, ab are

$$-\frac{(a'b)}{(a'c)},\ -\frac{(b'c)}{(b'a)},\ -\frac{(c'a)}{(c'b)}$$

respectively. The condition of collinearity of the points of division gives

$$\frac{(a'b)}{(a'c)}\times\frac{(b'c)}{(b'a)}\times\frac{(c'a)}{(c'b)}=1$$

But $(a'b)=(ba')$ etc.

Hence the above condition may be written as

$$\frac{(ab')}{(ac')} \times \frac{(bc')}{(ba')} \times \frac{(ca')}{(cb')} = 1;$$

and this is the condition for the collinearity of the conjugates of a, b, c lying on $b'c'$, $c'a'$, $a'b'$ respectively.

A similar argument gives the following theorem :

If a, b, c; a', b', c' be two triads of points in space, α, β, γ the conjugates of a', b', c' lying on bc, ca, ab respectively and $a\alpha, b\beta, c\gamma$ are concurrent, then if α', β', γ' be the conjugates of a, b, c lying on $b'c'$, $c'a'$, $a'b'$ respectively, $a'\alpha', b'\beta', c'\gamma'$ are also concurrent.

From this we also deduce that—

If a, b, c; a', b', c' be two sets of circles and the three circles α, β, γ coaxal with b, c; c, a; a, b and orthogonal to a', b', c' respectively, be such that a, α; b, β; c, γ have a common coaxal circle, then the three circles α', β', γ' coaxal with b', c'; c', a'; a' b' and orthogonal to a, b, c respectively are also such that a', α'; b', β'; c', γ' have a common coaxal circle.

As a particular case of the above theorem we get the following analogue of Steiner's theorem :

If two triangles ABC, $A'B'C'$ be such that the perpendiculars from A, B, C to $B'C'$, $C'A'$, $A'B'$ meet BC, CA, AB in collinear points, then the perpendiculars from A', B', C' to BC, CA, AB meet $B'C'$, $C'A'$, $A'B'$ also in collinear points.

These results can, of course, be easily extended to higher dimensions.

GLEANING

A girl who worked with a micrometer was explaining to a friend that the micrometer was accurate in very fine measurements, including thousandths of an inch.

"Thousandths of an inch? My, that's tiny," said the second girl. "How many of them are there in an inch?"

"I don't know," replied the first girl, "but there must be millions of them."

From: *The Mathematics Teacher.*

ON THE EQUATORIAL COORDINATES OF A STAR AND THE RECTANGULAR COORDINATES OF ITS IMAGE ON THE PHOTOGRAPHIC PLATE*

BY

S. A. HAMID, *Govt. College, Lahore, and Punjab University observatory.*

We start with the following Lemma which is easily proved.

1. LEMMA. If two great circles of a sphere intersect at right angles, so do their projections from the centre on to its tangent plane at any point on either of the two circles.

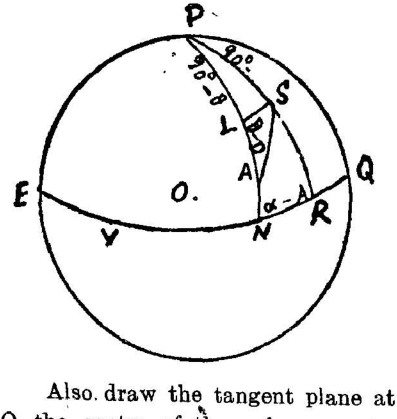
2. Let α, δ be the right-ascension and declination of a star S; A, D those of the point A on the celestial sphere, towards which the centre of the photographic plate is pointed. Also let ξ, η be the rectangular co-ordinates of the image of S on the photographic plate, referred to the plate centre as origin. Then we shall prove in this paper that

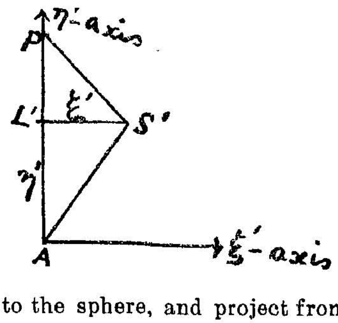
$$\xi = k \tan(\alpha - A) \sec(\theta - D) \cos\theta,$$

$$\eta = k \tan(\theta - D),$$

where $\tan\theta = \sec(\alpha - A)\tan\delta$, and give a meaning to k.

Draw the great circle SL perpendicular to the great circle AP. Join AS by an arc of a great circle. Let NL $=\theta$, then AL $=\theta - D$.





Also draw the tangent plane at A to the sphere, and project from O, the centre of the sphere, on to this plane. Let P', S', L' be the

* Smart: *Astronomy*, Arts. 160, 161.

projections of P, S, L respectively. The projection of AP will be AP′; of AS, AS′. Also AP′, AS′ will be the tangents at A to circles AP and AS. Thus by the definition of a spherical angle, $\angle$ PAS = $\angle$ P′AS′.

In the tangent plane, take AP′ as η'-axis, and a line perpendicular to AP′, drawn east-ward (the direction of right ascension increasing) as ξ'-axis.

By the Lemma, L′S′ $\perp$ L′A, so that AL′ = η', L′S′ = ξ'. Also AL′ = OA tan AL.

That is $\eta' = OA \tan(\theta - D)$ ... (2·1)

In the spherical triangles LAS, LPS, right-angled at L,

$$\tan PAS = \frac{\tan LS}{\sin AL}; \quad \tan APS = \frac{\tan LS}{\sin LP}$$

whence $\tan PAS = \dfrac{\tan(\alpha - A)\cos\theta}{\sin(\theta - D)}$ ... (2·2)

Again

$$\xi' = L'S' = AL' \tan P'AS'$$

$$= OA \tan(\theta - D)\,\frac{\tan(\alpha - A)\cos\theta}{\sin(\theta - D)},$$

by virtue of (2·1) and (2·2).

Hence $\xi' = OA \tan(\alpha - A)\sec(\theta - D)\cos\theta$ ... (2·3)

Again in the triangle LPS,

$$\cos APS = \frac{\tan LP}{\tan SP} = \frac{\cot\theta}{\cot\delta}$$

or

$$\cos(\alpha - A) = \frac{\tan\delta}{\tan\theta}$$

$\therefore$ $\tan\theta = \sec(\alpha - A)\tan\delta$ ... (2·4)

3. The centre O of the celestial sphere is supposed to be at the centre of the object glass of the telescope. If the projection of a star S on the tangent plane at A be S′ – with co-ordinates ξ' η' – and on the photographic plate, inserted in the focal plane of the telescope, S″ – with co-ordinates ξ, η, – the axes of ξ and η being respectively parallel to the axes of ξ' η', and drawn in the opposite sense, then the figures in the tangent plane at A and on the photographic plate are similar, the ratio of corresponding elements being OC/OA where OC is the focal length of the telescope. Hence from (2·3) and (2·1) we have

$$\xi = \xi' \, OC/OA = k \tan(\alpha - A)\sec(\theta - D)\cos\theta$$

$$\eta = \eta' \, OC/OA = k \tan(\theta - D)$$

where θ is given by (2·4) and k = OC is the focal length of the telescope in terms of the units of length on the photographic plate.

CONTRIBUTIONS TO THE THEORY OF THE CONSTRUCTION OF BALANCED INCOMPLETE BLOCK DESIGNS

BY

S. CHOWLA, *Government College, Lahore.*

1. Following Bose and Singer I have proved the following results:—

Let p be a prime congruent to 2 (mod. 3). Then we can find $p+1$ integers d_0, d_1, d_2......d_p such that

(i) The congruence $n \equiv (d_i - d_j) \bmod (p^2+p+1)$ has exactly one solution in i, j whenever n is not a multiple of p^2+p+1.

(ii) The residues (mod p^2+p+1) of the set

(S) $d_0, d_1, d_2, \ldots\ldots d_p$

are the same as the residues of the set

(S′) $pd_0, pd_1, pd_2, \ldots\ldots pd_p$

In the special case when $p=11$ the set of d's satisfying (i) and (ii) is

(S) 4, 6, 34, 40, 41, 44, 52, 61, 66, 85, 108, 124 (mod 133).

It is easily verified that every number is congruent (mod 133) to a difference of two numbers of the set (S), and that the residues of the set (S) multiplied by 11 are

(S′) 44, 66, 108, 41, 52, 85, 40, 6, 61, 4, 124, 34 (mod 133).

The numbers of the set (S′) are obviously a re-arrangement of the numbers of the set (S).

This result, when p is a prime congruent to 2 (mod 3), includes Singer's.

The following is a similar extension of R. C. Bose's theorem:—

Let p be a prime. We can find p integers $d_1, d_2, \ldots\ldots, d_p$ such that

(i) If n is not a multiple of $(p+1)$, the congruence $n \equiv (d_i - d_j)$ (mod p^2-1) has exactly one solution (in i, j).

(ii) The residues of the set

(S) $d_1, d_2, d_3, \ldots\ldots, d_p$

are the same as the residues (mod p^2-1) of the set

(S′) $pd_1, pd_2, pd_3, \ldots\ldots pd_p$

When $p=13$ the required set of d's is

(S) 6, 25, 29, 41, 78, 98, 103, 116, 142, 157, 163, 164, 166.

These numbers multiplied by 13, and reduced (mod 168) give the set

(S′) 78, 157, 41, 29, 6, 98, 163, 164, 166, 25, 103, 116, 142

which are the same as those of the set (S) in a different order.

2. We shall call the set S of numbers $d_1, d_2, \ldots\ldots, d_g$ a "difference-set" (mod n) where n is any positive integer, provided the number of solutions of the congruence

$$(2\cdot1) \qquad m \equiv d_i - d_j \pmod{n}$$

is independent of m whenever m is not a multiple of n. By the number of solutions of (2·1) we mean the number of distinct pairs (i, j) satisfying (2·1); i, j are naturally restricted to be integers between 1 and g, both included.

A theorem of R. C. Bose in *Annals of Eugenics*, Vol. 9, states that the quadratic residues of p, that are prime to p, form a "difference-set" (mod p) provided p is a prime congruent to 3 (mod 4). An example of this elegant theorem shows that the numbers

(S) 1, 4, 5, 6, 7, 9, 11, 16, 17

are a "difference-set" (mod 19). In fact, every number prime to 19 can be represented exactly 4 times as a difference (mod 19) of numbers of the set (S). Thus, for example 6 is congruent to $7-1$, $11-5$ $17-11$, $4-17$ (mod 19).

We now recall the familiar fact that if p is a prime congruent to 1 (mod 4), there are exactly $\frac{p-1}{4}$ *biquadratic residues* of p that are prime to p.

When do these biquadratic residues form a "difference-set" (mod p)? The answer is known to be in the affirmative when $p=37$ (R. C. Bose; *Annals of Eugenics* Vol. 9) and when $p=101$, 197 (Chowla: *Proc. Lahore Phil. Society*, Vol. 6, 1944, *pp.* 13·14).

R. C. Bose in a letter to the author dated 15—4—1944 conjectured that the biquadratic residues (mod p) form a difference-set (mod p) whenever $(p-1)/4$ is the square of an odd prime.

By using well-known results in the theory of cyclotomy (see Bachmann's *Die Lehre von der Kreistheilung*) I have succeeded in

proving the correctness of R. C. Bose's conjecture. I have further verified the result when p is 677.

3. *The isomorphism of R. C. Bose's geometrical solution of the design* $v=b=15$, $r=k=7$ *with Fisher and Yate's "cyclic" solution of the same.* The 'cyclic' solution is $a\ b\ c\ e\ f\ i\ k$ *i.e.*, the numbers.

$$1, 2, 3, 5, 6, 9, 11$$

taken (mod 15) [here we have an example of a difference-set (mod 15)]. Putting

$2=3'$	$5=5'$	$7=8'$
$3=1'$	$10=10'$	$11=4'$
$6=2'$	$15=15'$	$1=6'$
$12=11'$	$13=13'$	$8=12'$
$9=7'$	$14=14'$	$4=9'$

we see that the solution with the 'dashed' numerals is, on dropping the dashes, the same as R. C. Bose's geometrical solution of this design, given in *Annals of Eugenics* Vol 9.

4. I have proved the following

Theorem. Let m be equal to the product of p and $p+2$ where both p and $p+2$ are primes. We know there are exactly $(p^2-1)/2$ incongruent numbers x such that

$$\left(\frac{x}{p}\right)\left(\frac{x}{p+2}\right)=+1, \tag{4·1}$$

where the 'bracket' symbols on the left are Legendre symbols. Then, the above $(p^2-1)/2$ numbers x (each x is taken to lie between 0 and m), together with the p numbers between 1 and m (m included) which are multiples of $p+2$, give us in all

$$\frac{p^2-1}{2}+p=\frac{m-1}{2}$$

numbers, which form a difference set (mod m).

Example 1. $m=3·5=15$;

here 3 and 5 are both primes. The x which satisfy (4·1) are the numbers 1, 2, 4, 8.

These with the numbers 5, 10, 15 give us

$$1, 2, 4, 5, 8, 10, 15$$

which are a difference-set (mod 15). Adding 1 to each of these numbers we get the difference-set of para 3.

Example 2. $m = 5 \cdot 7 = 35$.

The reader will easily verify that we get a set of 17 (numbers which form a difference-set (mod 35).

5. I have also solved problems of which the following is typical.

91 persons (belonging to 7 different nations and 13 different professions) are invited out to 91 dinners each dinner consisting of 36 persons. Is this possible if

(*i*) *every* person is invited exactly 36 times;

(*ii*) *any two* people, *not belonging to the same nation or the same profession*, are invited equally often ?

The answer is 'yes'. The arrangement is possible.

(DEDICATED TO R. C. BOSE)

A PERFECT DIFFERENCE SET OF ORDER 18

BY

S. CHOWLA AND DALJIT SINGH.

All known perfect difference sets of order not exceeding 17 have been tabulated by Singer in *Trans. Amer. Math. Soc.* Vol. 43 (1938) p. 384. As far as we know a perfect difference set of order 18 has not been obtained so far. An example is given by the following 18 numbers:

0, 1, 3, 30, 37, 50, 55, 76, 98, 117, 129, 133, 157, 189, 199, 222, 293, 299.

Every integer prime to 307 can be obtained exactly once as the difference (mod 307) of two numbers of the above set of 18 integers.

GLEANING

The Predictor

is a huge camera-like instrument for tackling enemy planes. It has a number of sights or eye pieces. As soon as these are turned on the enemy plane, the instrument begins to work out its direction, height and velocity. From these it estimates how a shell should be fired to hit the enemy. It calculates not where he is now but where he will be when the shell reaches him. A cable goes from the predictor to the gun-man and he sets his guns looking not at the enemy plane but at the moving dial connected to the predictor.

Annamalainagar Math. Exposition

ON THE RADICAL CONIC OF TWO CENTRAL CONICS.

BY

V. RANGACHARIAR AND B. SINGH, *Patna.*

1. This paper extends the concepts of the "power of a point", and "radical axis" to conics, and obtains the properties of a "a co-axial pencil of conics," as generalisations of the corresponding properties of circles.

2. Let O be a point and $f(x, y) = 0$ a conic. If OPQ be a secant cutting the conic in P and Q, and CD be the parallel semi-diameter, then we have

$$\alpha\,\beta \cdot \frac{OP \cdot OQ}{CD^2} = \frac{f(x, y)}{\sqrt{(ab - h^2)}} = \frac{S}{\sqrt{C}} \qquad \dots \quad (1)$$

where O is the point (x, y) and α, β are the semi axes of the conic. We define the expressions in (1) to be the "*power of* O *with respect to the conic S.* If $OP_1\,Q_1$ and $OP_2\,Q_2$ be chords parallel to the principal axes, then obviously the power of O is $\sqrt{(OP_1 \cdot OQ_1 \cdot OP_2 \cdot OQ_2)}$.

3. If $S_1 = 0$ and $S_2 = 0$ be two given conics, the locus of points whose powers with respect to them are equal is $S_1/\sqrt{C_1} = S_2/\sqrt{C_2}$ where and $\sqrt{C_1}$, $\sqrt{C_2}$ are taken both positively. This locus is real only when the conics are both ellipses or both hyperbolas. *This conic is named the Radical Conic of the two given conics.*

4. Let us examine the condition that any two numbers of the pencil $S_1 + \lambda S_2 = 0$ may have the same Radical Conic. The Radical Conic of $S_1 = 0$ and $S_1 + \lambda S_2 = 0$ is given by

$$S_1/\sqrt{C_1} = (S_1 + \lambda S_2)/\sqrt{(\lambda^2 C_2 + \lambda A + C_1)} \qquad \dots \quad (2)$$

where $A \equiv a_1\,b_2 + a_2\,b_1 - 2h_1\,h_2$. If (2) be the same for all values of λ we must have $A \equiv 2\sqrt{C_1\,C_2}$. Hence "*If the two conics are such that an asymptote of one is parallel to an asymptote of the other, any pair of conics of the pencil $S_1 + \lambda S_2 = 0$ have the same Radical Conic the Radical Conic of $S_1 = 0$ and $S_2 = 0$.*" ... (3)

Such a pencil may be named a "co-axial pencil of conics". The parabolas of the pencil are given by $\lambda^2 C_2 + \lambda A + C_1 = 0$ i.e.,

$(\lambda\sqrt{C_2}+\sqrt{C_1})^2=0$ since $A=2\sqrt{C_1\,C_2}$. Hence "*The Radical Conic of the pencil is the parabola belonging to the pencil.*" ... (4)

5. Let $S_1=0$, $S_2=0$, $S_3=0$ be three given conics. Their Radical Conics in pairs are given by

$$\frac{S_1}{\sqrt{C_1}}=\frac{S_2}{\sqrt{C_2}}\,;\;\frac{S_2}{\sqrt{C_2}}=\frac{S_3}{\sqrt{C_3}}\,;\;\frac{S_3}{\sqrt{C_3}}=\frac{S_1}{\sqrt{C_1}}$$

which clearly intersect in four points. Hence "*Any three conics will have four Radical Centres*". ... (5)

6. We define the *limiting points* of a co-axial pencil to be the *centres of the degenerate members of the pencil.* Evidently there are three limiting points—the diagonal points of the quadrangle formed by the four common points. The three limiting points will be all real only when the intersections are all real or all imaginary. When the conics are such that only one asymptote of one is parallel to one asymptote of the other, three of the intersections are finite and one infinite. But the limit-points in this case are all finite and can be located, whereas, when the conics have both asymptotes parallel— as in the case of circles — two of the limit points are finite and one is at infinity.

7. In three dimensions we define the power of O with respect to a conicoid $F(x, y, z)=0$ as $(\alpha\beta\gamma)^{\frac{1}{3}}$ $OP\cdot OQ/CD^2=F(x, y, z)/\Delta^{\frac{2}{3}}$ where α, β, γ are the semi-axes of the quadric where Δ is the discrimmant of the homogeneous terms of the second degree. The Radical quadric of $S_1=0$ and $S_2=0$ is given by $S_1/\Delta_1^{1/3}=S_2/\Delta_2^{1/3}$.

If the pencil of quadrics $S_1+\lambda S_2=0$ have in pairs the same Radical Quadric, then the three paraboloids in the system all coincide and this is the radical quadric of the "*co-axial pencil of quadrics*". The condition for this is that the sections of S_1 and S_2 by the plane at infinity should have 3-point contact. "The Radical curve of three given quadrics is a quartic curve. "Four given quadrics have eight Radical Centres".

GLEANING

If one of our ancestors at Mohenjadaro had invested one pie at half percent compound interest our heritage will now amount to Rs. 50,000,000,000.

If the population of India (400 millions) were arranged in a line (allowing half a yard for each person), the line would extend over 100,000 miles, that is just about four times round the world or half way to the moon.

Annamalainagar Math. Exposition

TRANSCENDENCE OF DECIMALS

BY

AMRITSAGAR PURI, *Research Scholar, Punjab University.*

1. The paper establishes sufficient conditions for decimals in any scale p to represent transcendental numbers, i.e. numbers which do not satisfy an algebraic equation, with integral co-efficients.

2. Every positive number can be expressed as a decimal in the scale of p in the form

$$x = \mathrm{I} + \frac{a_1}{p} + \frac{a_2}{p^2} + \cdots \quad \ldots \quad (2\cdot1)$$

where I is a non-negative integer and every a is a non-negative integer less than p, conversely, every series of the type (2·1) converges to a positive number. Omitting the vanishing terms of (2·1) we have

$$x = \mathrm{I} + \frac{a_{q1}}{p^{q1}} + \frac{a_{q2}}{p^{q2}} + \cdots \quad \ldots \quad (2\cdot2)$$

For the considerations of this paper we suppose that $q_1, q_2, q_3, \ldots$ form an infinite sequence. We now prove the following theorem,

3. THEOREM 1. *If the infinite sequence* $(q_{n+1} - rq_n)$ *is not bounded, x cannot satisfy an algebraic equation, with integral co-efficients, of degree less than* $(r+1)$

Proof:—It will suffice to prove that x is not a root of such an equation of degree r. If possible, let x satisfy

$$b_r\ x^r + b_{r1}\ x^{r1} + \cdots + b_o = 0, \quad \ldots \quad (3\cdot1)$$

where the b's are integers and $r > r_1 > r_2 > r_3 \cdots$

We write $x = y + c$, where

$$y = \mathrm{I} + \frac{a_{q1}}{p^{q1}} + \frac{a_{q2}}{p^{q2}} + \cdots + \frac{a_{qn}}{p^{qn}} \quad \ldots \quad (3\cdot2)$$

and

$$c = \frac{a_{q_{n+1}}}{p^{q^{n+1}}} + + \cdots \leqslant 1/p^{\,q_{n+1}^{\;-1}} \quad \ldots \quad (3\cdot3)$$

We now have

$$b_s\ x^s = b_s\ (y+c)^s \div b_s\ y^s + c\ \xi_s, \qquad (s = r,\ r_1\ r_2 \cdots)\ \xi_o = 0$$

Clearly $|\xi_s| = |b_s|\ (sC_1\ y^{s-1} + sC_2\ y^{s-2}c + \cdots\cdots)$ ... (3·4)

$\leqslant |b_s|\ (sC_1\ y^{s-1} + sC_2\ y^{s-2} + \cdots\cdots)$ since $(c<1)$

$< |b_s|\ (1+x)^r < \xi_s$ say (independent of n)

$s = r,\ r_1 \cdots\cdots$

(3·1) can be rewritten

$$\sum_{s=0,\,r,\,r_1,\,\cdots} b_s\, y^s + c \sum_{s=r,\,r_1,\,\cdots} \xi_s = 0. \qquad \text{... (3·5)}$$

We then get

$$\left| \sum_{s=0,\,r,\,r_1,\,\cdots} b_s\, y^s \right| \geqslant 1/p^{rq_n}$$

and $\left| c \sum_{s=r,\,r_1,\,\cdots} \xi_s \right| < cr\,\xi \leqslant \dfrac{r\,\xi}{p^{q_{n+1}-1}} < \dfrac{1}{p^{q_{n+1}-1-\eta}}$,

if η is such that $r\xi < p^{\eta}$, η being also independent of n.

If $q_{n+1} - rq_n > 1 + \eta$, (and this is true for an infinite number of n's, since the sequence $q_2 - rq_1$, $q_3 - rq_2$, $\cdots\cdots$ is unbounded) then (3·5) is not true and this is a contradiction. Hence the theorem is proved.

§ 4. THEOREM 2. *If the infinite sequence* $\frac{q_2}{q_1}, \frac{q_3}{q_2}, \frac{q_4}{q_3}, \cdots\cdots$ *is not bounded, then* x *is transcendental.*

Theorem 2 is an easy corollary of theorem 1. Whatever be r, the sequence $q_2 - rq_1$, $q_3 - rq_2 \cdots\cdots$ is unbounded. Thus x does not satisfy any rth degree equation. whatever r may be.

§ 5. **Examples.** The number

$$\frac{1}{p} + \frac{1}{p^4} + \frac{1}{p^{27}} + \frac{1}{p^{256}} + \cdots\cdots,$$

where and $p \geqslant 2$ the successive powers of p are 1^1, 2^2, 3^3, $4^4 \cdots\cdots$ is transcendental. The number

$$\frac{1}{p} + \frac{1}{p^4} + \frac{1}{p^{14}} + \frac{1}{p^{45}} + \frac{1}{p^{139}} + \cdots\cdots,$$

where $4 = 1 \cdot 3 + 1$, $14 = 4 \cdot 3 + 2$, $45 = 14 \cdot 3 + 3$, etc. and $p > 1$ is not an algebraic number of degree less than 4.

The number $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^7} + \frac{1}{2^{11}} + \frac{1}{2^{16}}, \cdots\cdots$ is irrational. Such examples can easily be multiplied.

§ 6. We have the series

$$\frac{a_1}{u_1} + \frac{a_2}{u_2} + \frac{a_3}{u_3} + \cdots\cdots + \frac{a_n}{u_n} + \cdots\cdots \qquad \text{... (6·1)}$$

in which $a_1, a_2 \ldots$ are non-negative integers which are bounded and u_1, u_2 are non-zero positive integers such that the L. C. M. of $u_1, u_2, \ldots, u_{n-1}$, divides u_n for all n. We form the series

$$\frac{1}{2^{p_1}}+\frac{1}{2^{p_2}}+\cdots+\frac{1}{2^{p_n}}+\frac{1}{2^{p_{n+1}}}+\cdots \qquad \ldots (6\cdot 2)$$

by defining p_n as follows: $2^{p_n} \leqslant u_n \leqslant 2^{p_{n+1}}$.

THEOREM 3. *If the sequence* $\left(\frac{p_{n+1}}{p_n}\right)$ *is unbounded, then* (6·1) *is transcendental; if the sequence* $(p_{n+1}-rp_n)$ *is bounded,* (6·1) *is not an algebraic number of degree less than* $(r+1)$.

The proof is similar to that of theorem 1.

Ex. 1. $\sum \frac{1}{2^n!} = \frac{1}{2!}+\frac{1}{(2^2)!}+\cdots+\frac{1}{(2^n)!}+\frac{1}{(2^{n+1})!}+\cdots$ is transcendental, for

$$(2^{n+1})! = 2^{n+1}(2^{n+1}-1)\cdots(2^n+1)(2^n)! > 2^{n2^n}(2^n)! \geqslant 2^{p_n+n^{2^n}}$$

$$\therefore \quad \frac{p_{n+1}}{p_n} \geqslant 1+\frac{n2^n}{p_n} > n. \quad \text{Hence the result.}$$

GLEANINGS

Origin Of The Word Academy

The First *Academy* was a pleasure garden in Athens which belonged to one *Academus*, walled by Hipparchus. Here Plato taught for 50 Years and the Academy thus started lasted for 300 Years *i.e.* from about 400 B. C to 100 B. C. All subjects were discussed; Arts, Science, Letters, Mathematics, Philosophy.

The Royal Asiatic Society (est. 1784; Patron: Warren Hastings) is the earliest Indian Academy. Later once are: U. P. Academy of Sciences (1930) later renamed the *National Academy of Sciences* and *The Indian Academy of Sciences, Bangalore* (1934).

* * * *

Comparisons

At one anna per 1000 miles the fare

	Rs.	As.	Ps.
To go Round The Earth (25,000 m.) is	1	9	0
Fare to go to the Moon (238,000 m.) is	14	7	0
" " to the Sun (93,000,000 m.) is	5,812	0	0
" " to the nearest Star (26,000,000,000,000 m.) is over	Rs. 1500,000,000		

* * * *

Scale 1 inch = 10,000 light Years.
Earth: an electron
Naked-eye Star Sphere: 3/5″ diameter
Galaxy (greater milky way) 20″ diameter
Diameter of telescopic range: 20,000″ 1/3 mile.

Annamalainagar Math. Exposition

ON THE METRIC GEOMETRY OF A CYCLIC N-POINT—I

BY

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Can we extend the familiar properties of a triangle to a figure with n vertices? It sounds hopeless in view of the absence of various circles, and of the number of joins of the vertices being different from that of the vertices except for $n=3$. In this paper it is shown, however, that if we artificially endow our n-point with a circumcircle by making all the points concyclic, then it is possible to associate with each set of r points a "central line" symmetrically related to all of them, which for $r=2$ becomes their join and for $r=n-1$ plays the same role in the metric geometry of the n-point as the sides do in a triangle. With the help of these central lines are obtained remarkable analogues to properties associated with the orthocentre, 9-points circle, polar circle etc. in the case of a triangle.

1. The orthocentre of a cyclic n-point.

Let $T_1, T_2, \cdots T_n$ be a set of n real points on the circumference of the unit circle, and let the *turns* * which correspond to them be $t_1, t_2, \cdots t_n$ so that their absolute coordinates † are

$$x=t_r, \qquad y=t_r^{-1}. \quad (r=1, 2, \cdots n).$$

The circumcentre of the cyclic n-point $T_1 \cdots T_n$ is the origin T, and their centroid G is the point

$$x=(t_1+t_2+\cdots\cdots t_n)/n, \tag{1·1}$$

which implies the associated equation

$$y=(t_1^{-1}+t_2^{-1}+\cdots t_n^{-1})/n.$$

We *define* the orthocentre H of the n-point as the point obtained by joining TG and producing it in the ratio $1:n$ (n being the number of vertices), so that the orthocentre is given by

$$x=t_1+t_2+\cdots t_n \tag{1·2}$$

Remembering that the x-coordinate of a point is the complex number which names it in the Gauss plane and hence obeys the law of

* "Turn" is a convenient word due, I believe, to Prof. Morley to denote a complex number whose modulus is unity.

† The "Absolute coordinates" of a point are $x=X+iY$ and $y=X-iY$ where X,Y are cartesian rectangular coordinates. They are conjugate imaginaries if the point be real.

vector addition, we deduce that the vector TH (from the circumcentre to the orthocentre) is the sum of the vectors TT_1, TT_2, $\cdots$ TT_n. (1·3)

The line joining the circum and orthocentres will be called the *Euler line* of the n-point and any point E on it an *Euler Centre.* The circle with its centre at E and whose radius bears to that of the circumcircle the ratio $k=\text{TE}/\text{TH}$ is the *Euler circle of E.* Thus the Euler circle of H, (the orthocentric circle), has its centre at H and its radius 1 equal to that of the circumcircle. It is clear that any Euler circle may be obtained from the orthocentric circle by a uniform dilatation or contraction from T in the ratio $1:k$. In particular, for $k=\frac{1}{2}$ we have the *medial centre* M, the mid-point of TH, and the *medial circle* whose radius is half that of the circumcircle.

The locus of points equidistant from the circum and orthocentres is a line called the *central line* of the n-point. It will be found that, in the extension of the familiar properties of the triangle to any set of concyclic points, the central line of subsets plays the same role as the sides of a triangle. It follows from the definition that the central line of a polygon passes through its medial centre and is perpendicular to its Euler line. (1·4)

The orthocentre of a single point on an assigned circle is itself, while its Euler line is the radius through it. For a pair of points T_1, T_2, the orthocentre is the reflection of the circumcentre T in the chord joining them and the Euler-line is the diameter bisecting the chord. The central line of a 2-point is their join. For the inscribed triangle the nomenclature is the same as that ordinarily in use, except that the medial centre and circle are more generally known as the nine-point centre and nine-points circle of the triangle.

2. The Coolidge Chain.

It is now easy to prove analytically a chain of theorems due to Prof. Coolidge* showing the incidence relations between the Euler-centres and circles of a cyclic n-point and of those derived from it by the omission of one of its points.

Consider the Euler centre $x=k\ (t_1+t_2+\cdots t_{r-1}+t_{r+1}\cdots t_n)$ of the $n-1$ point $T_1T_2 \cdots T_{r-1}\ T_{r+1} \cdots T_n$ corresponding to any ratio $k:1$. Symmetrizing this for n points by writing $s=t_1+t^2+\cdots t_n$ we have $x=ks-kt_r$ which for all values of r lies on the circle

* Some circles associated with concyclic points, *Annals of Mathematics,* Vol. 12 (1910) page 40-42 where the theorems are proved primarily for the medial centre and the medial circle, and may be extended to any Euler circle or centre by a uniform dilatation from the circumcentre.

$x = ks - kt$ where t is a variable turn; but this has for centre the Euler centre ks of the n-point and k for radius and is, by definition an Euler circle.

Again the Euler circle of $T_1T_2 \cdots T_{r-1}T_{r+1} \cdots T_n$ is given by

$$x = k(t_1 + t_2 + \cdots t_{r-1} + t_{r+1} + \cdots t_n) + kt = ks + k(t - t_r)$$

which for $t = t_r$ passes through the Euler centre $x = ks$ of $T_1 \cdots T_n$.

Hence:

The Euler centres of each of the $n-1$ point sets of a cyclic n-point lie on the Euler circle of the n-point. Also the Euler circles of each of the n-1 point sets pass through the Euler centre of the n-point. (2·1)

It should be noted that the Euler circles and centres form a double chain. Thus if we start by associating a *point*, say the orthocentre ($k = 1$) with a triangle, we shall have a circle associated with 4 points on which lie the orthocentres of the 4 derived 3-points; a point associated with 5 concyclic points *viz* the point of intersection of 5 such orthocentric circles, and so on, a point for n odd and a circle for n even. On the other hand, if we start with a circle associated with a 3-point namely the circle with the orthocentre for centre and radius equal to that of the circumcircle, we shall have a similar chain of alternating points and circles, but a point will now be associated with an even number of concyclic points and a circle with an odd number, with incidence relations of the same kind as before. The point associated with a set of concyclic points in one chain is the centre of the circle associated with it in the other.

3. The close analogy which the above configuration presents to that of the centre circles of a system of n lines* is explained by the fact that we may associate with a cyclic n-point a system of n lines whose centre circles are the Euler-circles of the n-point.

We find such a configuration of lines by means of the "deltoid" (three cusped hypocycloid) which is the envelope of the lines

$$xt^2 - yt = k\ (t^3 - 1) \qquad (3·1)$$

t being a variable *turn*. Two lines t_1 and t_2 intersect in the point

$$x = k\ (t_1 + t_2 + 1/t_1t_2) = k(s_1 - t_3 + t_3/s_3)$$

whence, the equation of the circumcircle is

$$x = k(s_1 - t + t/s_3)$$

s_1 and s_3 standing for the sum and product of t_1, t_2, t_3.

* attributed by Prof. Coolidge to Pesci: *Periodico di Mathematica* Vol. 5 (1891).

Hence the circumcentre of the triangle formed by the lines t_1, t_2, t_3 is the Euler-centre of the triad of points t_1, t_2, t_3 on $x = t$ for the ratio $k : 1$. With a fourth tangent t to the deltoid we have a system of 4 lines whose circumcentres when the 4 lines are taken in sets of three, are concyclic since these are the Euler centres of triads formed out of t_1, t_2, t_3, t_4, and the centre of this circle is the "centre" associated with the 4-line and is equally the Euler centre of the 4-point. It is thus seen that the Pesci and Coolidge chains are not independent of each other.

4. Metrical properties of the Orthocentre.

If we denote by h_r the orthocentre of the n-1 points obtained by omitting T_r, TH is by (1·3) the vector sum of Th_r and TT_r, whence we deduce that Th_r is parallel to HT_r. Combining this with (1·4) we see that

> the orthocentre is the point of concurrence of the perpendiculars dropped from each of the n-points T_r on their opposite central lines (*i.e.* the central lines of the remaining n-1 points) (4·1)

More generally, dividing the group of n points into two complementary sets we have the result:

> the line joining the orthocentre of the whole set to the orthocentre of any subset is perpendicular to the central line of the complementary set, (4·2)

a theorem which includes (4·1) since the orthocentre of a single point is itself. The result (4·2) will remain true if in the theorem we replace the orthocentre by the medial or any Euler centre provided the same value of k is retained throughout.

The following properties are also easily proved and offer striking anologies to the corresponding properties for a triangle.

> The distance of the orthocentre from a vertex is twice the distance of the circumcentre from the opposite central line. (4·4)

> The reflection of the orthocentre in the n central lines of subsets taken n-1 at a time all lie on the circumcircle. (4·5)

> The line joining the orthocentres of any complementary subsets is bisected at the medial centre of the whole set (4·6)

5. The Medial Circle.

The equation of the central line of $T_1T_2 \cdot \cdot T_{n-1}$ is

$$x \ \Sigma t_r^{-1} + y\Sigma t_r = \Sigma t_r \Sigma t_r^{-1} \quad (5\cdot 1)$$

the summation extending over $r=1,2...n.$ $(n\cdot 1)$

The perpendicular to (5·1) through T_n is

$$x\Sigma t_r^{-1} - y\Sigma t_r = t_n \Sigma t_r^{-1} - t_n^{-1}\Sigma t_r \quad (5\cdot 2)$$

the summations being the same as before. These two lines meet in

$$x = \frac{1}{2}s - \frac{t_1 + \cdot \cdot \cdot \cdot \cdot t_{n-1}}{t_1^{-1} + \cdot \cdot \cdot \cdot t_{n-1}^{-1}} \cdot \frac{1}{2t_n} \quad (5\cdot 3)$$

where $s = t_1 + \cdot \cdot \cdot t_r$. Now the second term on the right is the reciprocal of its conjugate and is hence a *turn*. Hence the feet of the perpendiculars from each of the vertices on the "opposite" central lines (*i.e.* the central lines of the remaining points) lie on

$$x = \frac{1}{2}s - \frac{1}{2}t$$

which represents the medial circle of the n-point.

Again, since the medial centre M is the mid-point of TH, it is equidistant from the projections of T and H on any line and in particular on the n central lines. Hence the medial circle passes also through the feet of the perpendiculars from T on the n-central lines *i.e.* through the mid-points of the central line chords.

Lastly the medial circle and circumcircle have H for their external centre of similitude and the ratio of their radii is 1 : 2. Hence the medial circle passes through the mid-points of HT_1, HT_2, $\cdot \cdot \cdot HT_n$. Combining all these we have the following analogue of the 9-pt. circle:

The medial circle of a cyclic n-point passes through the following $3n$ points:

(i) the feet of the perpendiculars from the vertices on their opposite central lines;

(ii) the mid-points of the central line chords;

(iii) the mid-points of the lines joining the orthocentre to each of the n vertices.

The centre of the circle is the mid-point of the line joining the orthocentre and circumcentre and its radius is half that of the circumcircle.

6. The Polar Circle.

We shall now show that

there exists a circle with respect to which each point and its opposite central line are pole and polar; its centre is H and its radius $\sqrt{(TH^2 - R^2)/2}$

A circle with its centre at the orthocentre and radius ρ is given by $x = s + \rho t$. The inverse of t_n with respect to this circle is

$$x = s + \frac{\rho^2}{\bar{t}_n - \bar{s}} = s - \frac{\rho^2}{t_1^{-1} + \cdots t_{n-1}^{-1}} \tag{6·1}$$

where $s = t_1 + \cdots t_n$ and $\bar{s}$, $\bar{t}_n$ are the conjugates of s and t_n. The point will be identical with the foot of the perpendicular from t_n on its opposite central line if the corresponding complex numbers, namely those given by (6·1) and (5·3) are equal. On simplifying, this is found to lead to

$$\rho^2 = \frac{1}{2}\left(\sum_1^n t \sum_1^n t^{-1} - 1\right) = \frac{1}{2}\left(TH^2 - R^2\right)$$

where R is the circumradius. This value being symmetric in $t_1 t_2, \cdots t_n$ will serve equally for all the vertices and their opposite central lines.

7. The Central lines.

Since the feet of the perpendiculars from T and H on the n central lines lie on a circle—the medial circle—it follows that the central lines of any cyclic polygon are tangents to a conic with T and H for foci. This conic has therefore its centre at M and the medial circle for auxiliary circle. These properties characterise the central lines which in their turn determine the cyclic polygon but not uniquely.

In fact any set of n lines all touching a conic are necessarily central lines of one of two sets of concyclic points T_r and H_r obtained by dropping perpendiculars from the foci T and H of the conic on them. The two sets of points thus obtained have the following properties :

(i) the orthocentre of each set is the circumcentre of the other; so that M the mid-point of TH is the common medial centre of both;

(ii) the orthocentre of any subset $T_1 \cdots T_{r-1} T_{r+1} \cdots T_n$ is H_r and the orthocentre of $H_1 \cdots H_{r-1} H_{r+1} \cdots H_n$ is T_r

(iii) More generally, the orthocentre of $T_1 \cdots T_r$ is also the orthocentre of $H_{r+1} \cdots H_n$.

8. Relation to Konnully's Orthocentric Chain.

In an article on the "orthocentre of a cyclic polygon" published in *The Mathematics Student* Vol XI pp 28-31, Rev. Augustine O. Konnully, obtains the following chain: Three concyclic points have an orthocentre. Given four concyclic points, the joins of each point with the orthocentre of the other three are concurrent at say the "orthocentre" of the 4-pt. With 5 points the joins of each point with the "orthocentre" of the remaining four as previously defined are concurrent at the "orthocentre of the 5-point" and so on.

With our notation the points successively obtained by the above construction are $(t_1+t_2+t_3)$, $(t_1+t_2+t_3+t_4)/2, \cdots (t_1+ \cdots t_n)/(n-2)$ and are therefore not the orthocentres of the cyclic n-points as defined in this paper except for $n=3$.

Since $t_1, t_2, \cdots$ are vectors from the circumcentre T to the vertices, it follows that $(t_1+\cdots t_n)/(n-2)$ is the centroid of unit masses at the vertices and a fixed mass -2 at the origin T, and the chain is a consequence of the well known property of the centroid that it is collinear with those of two complementary subsets. For the chain property, it is not necessary to have the points on a circle, nor to have the mass at the fixed point to be -2. Conversely, it is easy to show that every chain in which a point P_n is associated symmetrically with n points $A_1 \cdots A_n$ such that when an additional point A_{n+1} is introduced the associated P_{n+1} is collinear with every A_r and the associate of the remaining n points, must be a centroidal chain. In other words, the points P_n are the centroids of unit masses at the n-points and a fixed mass k positive or negative at a fixed point.

9. Higher orthocentres.

The concept of "orthocentre" defined in this paper may be extended to "higher orthocentres" by defining them by the complex numbers $\sum t$, $\sum t_1t_2/\sum t$, $\sum t_1t_2t_3/\sum t_1t_2$ etc. or convenient numerical multiples of these. It is easy to see that the n points thus defined are covariantly related to the n vertices under the group of rigid movements, reflections and similarity transformations and that together with the circumcentre they form a complete system. The study of these must however await another paper.

ON THE CLIFFORD AND GRACE CHAINS

BY

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1·0. The object of this expository article is to introduce to the young student of Mathematics a very interesting chain of theorems in geometry leading to an extra-ordinarily beautiful configuration of points and circles associated with the names of Miquel (1836), W. K. Clifford (1870), and J. H. Grace (1898).

The Grace Chain

2·0. The raw material of this chain is a set of given circles (1) (2) (3) . . . (n) all passing through a point which our notation requires to be nameless, and which we accordingly denote by ( ).

2·1. The first link of our chain of theorems is very simple and asserts that any pair of circles say (1) and (2) have, besides ( ), another intersection which we denote indifferently by (12) or 21)*. With three circles (1) (2) (3) through ( ), we have three points (12), (23), (31). Our second link simply affirms that they lie on a circle, which we shall denote by (123).

2·2. Starting with four circles (1), (2), (3), (4) through ( ), we shall have four circles (123), (234), (341), (412) which, the next theorem of our chain asserts, are concurrent at a point which we shall denote by (1234).

2·3. To prove this, we need only take the inverse of the whole figure with respect to a circle C with centre ( ). [(i.e.) replace every point in our figure by its inverse w. r. t. the circle C.] The circles (1) (2) (3) (4) now become straight lines (1)′ (2)′ (3)′ (4)′ and the circles (123) . . . . become the circumcircles of the triangles formed by the lines taken three at a time. Take two of these circles say (123)′ and (234)′. They pass through (23)′ and have another intersection say P. By the Simson line theorem, the feet of the perpendiculars from P on (1)′ (2)′ (3)′ are collinear as also those on (2)′ (3)′ (4)′. Hence, the feet of the perpendiculars on all the four lines are 'collinear† ' and, by the converse of the Simson line theorem, the circles (341)′ and (412)′ pass though P which is the inverse of our point (1234).

* Throughout this paper. the order of the letters occuring in any bracket symbol is a matter of indifference.

† P is the focus of the unique parabola touching the lines 1, 2, 3, 4.

3·0. Notice the special features of our way of naming points and circles. A good notation is to the theorems expressed in it what the body is to the soul. It must reflect faithfully the inner harmonies, and not only permit freely the developments latent within, but even suggest the directions along which growth and progress would be most natural. Since we shall deal with concurrent circles and concyclic points, our notation has been designed so that we can at once say whether a symbol refers to a point or a circle, and whether a point does, or does not, lie on a particular circle.

3·1. A bracket symbol with an even number (including zero) of letters represents a point, and one with an odd number represents a circle. A point lies on a circle if the symbol for the circle can be obtained from that of the point by the omission or the addition of a single letter. Thus the points lying on the circle (123) are (12), (23), (31) and (1234). If we make the convention that a letter repeated twice in a symbol may be suppressed, we can say that the points lying on (123) are (1231) (1232) (1233) and (1234). More generally the points lying on any circle with symbol (R)—where (R) is any combination of an odd number of letters (repeated or not)—are (R1) (R2) (R3) . . . (R*n*). Similarly the circles passing through any point, say (12), are (121) (122) (123) . . .

4·0. With five circles (1) (2) (3) (4) and (5) through ( ), we shall have five points like (1234). The next theorem of our chain asserts that they all lie on a circle which we shall denote by (12345).

4·1. To prove this, we have only to apply the result of § (2·2) to the four circles (123) (124) (125) and (121) = (2) passing through (12). The intersections of the first three circles in pairs other than (12) are (1234) (1245) and (1253). Let the circle through these be called (12345). To justify this name, we have to show that it passes also through (1345) and (2345). Now, the intersections of (124), (125) and (121) in pairs are (1245), (1251) = (25) and (1214) = (24), the circle through which is (245). Similarly from (123), (125) and (121) we derive the circle (235); and from (125), (121) and (124) the circle (234). Hence by the four circle theorem of §(2·2) the circles (12345), (234), (235) and (245) are concurrent at a point which can only be (2345) common to the last three circles. Likewise, starting from the four circles (123) (124) (125) and (122) = (1) through (12), we can prove that (12345) passes also though (1345)—thus justifying our calling it (12345), and proving the five circle theorem—the next link in the chain.

4·2. Generally, the n-circle theorem follows by applying the $\overline{n-1}$ circle chain to the circles through (12) say (121) (123) . . . (12n) and also to (122) (123) . . . (12n). Thus, the chain of theorems is seen to continue indefinitely.

5·0. The entire configuration (the Grace configuration) thus generated by n concurrent circles—like (1) (2) . . . (n)—consists of 2^{n-1} points represented by, all the even combinations of the n letters e.g. ( ), (i j), (i j k l), . . . and 2^{n-1} circles represented by all the odd combinations of the letters e.g. (i), (i j k), . . . each circle (c) passing through the n points (c1) (c2) . . . (cn) and each point P lying on the n circles (P1) (P2) . . (Pn). The configuration can be generated equally from the n concurrent circles through any of the 2^{n-1} points of the configuration.

The Clifford-Kantor Chain

6·0. Now, let (1) (2) (3) . . . (n) be a series of coplanar lines. Let (i, j) be the intersection of the lines (i) and (j); and let (i j k) be the circle through (ij) (jk) and (ki). Invert w.r.t. any circle with centre say at ( ), and let the lines (i) invert into circles (i)′ through ( ). Then (i j) will invert into the 'other intersection' (i j)′ of the circles (i)′ and (j)′; and the circle (i j k) will invert into the circle (i j k)′ through (i j)′ (j k)′ and (ki)′. Now, we have already proved §§ (2·2) (4·1) and (4·2) that the four circles (i j k)′(j k l)′(k l i)′(l i j)′ pass through a common point (i j k l)′ and that the five points (i j k l)′ (j k l m)′ (k l m i)′ (l m i j)′ and (m i j k)′ lie on a circle (i j k l m)′ and so on. Hence, in the original figure we get the

6·1. *Clifford-Kantor Chain of theorems*:

(6·12) Given two straight lines (1) (2), they have a common point (12).

(6·13) With three lines (1) (2) (3), the three points (12) (23) (31) lie on a circle (123).

(6·14) With four lines (1) (2) (3) (4) the four circles (123) . . pass through a common point (1234).

(6·1n) With n lines (1) (2) . . . (n) the n points (circles) (12 . . . $\overline{1r-}$ $\overline{r+1}$ . . n) lie on a circle (pass through a point) (12 . . . n).

7·0. The result (6·14) was first enunciated by Wallace in (1804) and (6·15) by Miquel (1836). That these incidences continue

indefinitely was proved by Clifford§ in (1870) and independently by Kantor§§ in 1878—by using the properties of generalised parabolas. For a very elegant and simple analytical proof, reference may be made to a paper by V. Ramaswami Iyer and M. Bhimasena Rao.†

7·1. That the configuration gains in symmetry on inversion was first pointed out by J. H. Grace †† after whom accordingly, we have named the chain of §(4·2). and the configuration of (5·0).

For further information on the subject, the reader may consult the following

References

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§§ *Weiner Berichte* Band 78. Uber line gattung etc.

† Miquel points and circles and centre-circles of a system of lines: *Jour. Ind. Math Society.* Vol. 16. pp. 270-8.

†† Circles, Spheres and Linear Complexes: *Trans. Camb. Phil. Society* 1898. 16. p. 31.

GLEANINGS

I know of scarcely anything so apt to impress the imagination as the wonderful form of cosmic order expressed by the " Law of frequency of Error". The law would have been personified by the Greeks and deified, if they had known of it. It reigns in serenity and in complete self-effacement amidst the wildest confusion. The huger the mob and the greater the apparent anarchy, the more perfect is its sway. It is the supreme Law of Unreason. Whenever a large sample of chaotic elements are taken in hand and marshalled in the order of their magnitude, an unsuspected and beautiful form of regularity proves to have been latent all along.

SIR FRANCIS GALTON: in *Natural inheritance.*

The more advanced the sciences have become, the more they have tended to enter the domain of mathematics, which is a sort of centre towards which they converge. We can judge of the perfection to which a science has come by the facility more or less great, with which it may be apprpoched by calculation.

QUETELET

NOTES AND DISCUSSIONS

Remarks on two notes in Vol. XI

The recent issue of the *Mathematics Student*, (Vol. XI, Nos. 1 & 2) contains several interesting discussions which are well within the scope of undergraduate mathematics. I am therefore tempted to offer some comments on two notes published there; (1) Dhamija's "Explanation of a Geometrical Fallacy" and (2) Ramanathan's "Trigonometrical Identities".

1. Dhamija's explanation furnishes a good example of 'analytical geometry annihilating geometrical intuition', a remark due to Prof. A. A. Krishnaswami Ayyangar. In the figure on page 48 of Dhamija's Note the point O is obviously the circumcentre of the triangle CDE; hence the perpendicular bisector of CE which passes through B, (BC = BE) and crosses BE must certainly meet HF produced in a point below K. This explains the fallacy without any elaborate analytical scaffolding. It may be remarked also that the *reductio ad absurdum* of a right angle being equal to an obtuse angle implies that the point O has been taken on the wrong side of K. The rule of the road has to be observed as much by mathematicians as by pedestrians.

2. In my amateurish mathematical wanderings I have sometimes come across some mathematical results obtained after a long labour, only to be suddenly disillusioned by the existence of a short cut confirming the triviality of my adventure. I was therefore highly interested in the Trigonometrical Identities on pages 40, 41, 42 which can be obtained so simply by direct trigonometrical methods. It is rather unfortunate that Ramanathan's rambles have led him to some mistakes in sign which vitiate his results (11), (13), (14), (15) and (16).

The following is a direct proof of the Trigonometrical Identities of the said Note:

$$\frac{\sin 3\phi}{\sin \phi} = \frac{\sin \phi \cos 2\phi + \cos \phi \sin 2\phi}{\sin \phi}$$

$$= \cos 2\phi + 2 \cos^2\phi = 1 + 2 \cos 2\phi \qquad \ldots \quad (1)$$

Putting $\psi = \theta/2$, $3\theta/2$, etc., in (1) and multiplying all the results, we get

$$(1+2\cos\theta)(1+2\cos 3\theta)\cdots(1+2\cos 3^{n-1}\theta)=\sin 3^n\frac{\theta}{2}\Big/\sin\frac{\theta}{2}$$

Changing ϕ to $\left(\frac{\pi}{2}-\phi\right)$ in (1), we get $-\dfrac{\cos 3\phi}{\cos\phi}=1-2\cos 2\phi$.

Giving values to ϕ and multiplying as before we get

$$(1-2\cos\theta)(1-2\cos 3\theta)\cdots(1-2\cos 3^{n-1}\theta)=(-1)^n\cos 3^n\frac{\theta}{2}\Big/\cos\frac{\theta}{2}$$

The factor $(-1)^n$ is omitted in Ramanathan's Note and hence his further identities based on this have become wrong.

Again, it is easy to verify that

$$\frac{\sin\theta}{1+2\cos\theta}=\frac{1}{4}\left\{\cot\frac{\theta}{2}-3\cot\frac{3\theta}{2}\right\}$$

and

$$\frac{\sin\theta}{1-2\cos\theta}=\frac{1}{4}\left\{\tan\frac{\theta}{2}-3\tan\frac{3\theta}{2}\right\}$$

which lead at once to Ramanathan's identities (12) and (13) with the addition of a negative sign on the right hand side of the latter.

Identity No. (15) of Ramanathan's Note when corrected becomes very trivial* involving term by term equality.

His identities (14) and (16) should be corrected to

$$\frac{\sin\theta}{1-4\cos^2\theta}+\frac{3\sin 3\theta}{1-4\cos^2 3\theta}+\cdots\cdots+\frac{3^{n-1}\sin 3^{n-1}\theta}{1-4\cos^2 3^{n-1}\theta}$$
$$=\frac{1}{4}\left\{\operatorname{cosec}\theta-3^n\operatorname{cosec}3^n\theta\right\}$$

Identities (21) and (23) readily follow from

$$\frac{1-2\sin\phi}{1+2\sin\phi}=\tan\left(\frac{\pi}{4}-\frac{\phi}{2}\right)\tan\left(\frac{\pi}{4}-\frac{3\phi}{2}\right)$$

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Some Properties of the Eulerian and Bernoullian Numbers

In this note we establish the relation between the Eulerian and Bernoullian numbers :—

We know that with the usual notation *

$$\sec x=1+\sum_{1}^{\infty}(-1)^n E_{2n}\frac{x^{2n}}{(2n)!};$$

$$x\cot x=1-\frac{B_2}{2!}(2x)^2+\frac{B_4}{4!}(2x)^4-\frac{B_6}{6!}(2x)^6+\cdots$$

* Vide Knopp's *Infinite Series.*

Substituting these in the identity $x \sec x = x \cot x \dfrac{d}{dx}(\sec x)$ we get

$$\sum_{n=0}^{\infty} (-1)^n E_{2n} \frac{x^{2n+1}}{(2n)!} = \left\{1 + \sum_{n=1}^{\infty} (-1)^n \frac{B_{2n}}{(2n)!} (2x)^{2n}\right\} \times \left\{\sum_{n=0}^{\infty} (-1)^n \frac{E_{2n}}{(2n-1)!} x^{2n-1}\right\}$$

Equating coefficients of x^{2n+1} on both sides we get;

$$\frac{E_{2n+2}}{(2n+1)!} + \frac{E_{2n}}{(2n-1)!}\left(\frac{1}{2n} + 2^2 \frac{B_2}{2!}\right) + \frac{B_4}{4!} 2^4 \frac{E_{2n-2}}{(2n-3)!} \cdots + \frac{B_{2n}}{(2n)!} 2^{2n} E_2 = 0$$

from which eliminating the Eulerian numbers we get

$$\frac{E_{2n+2}}{(2n+1)!} = (-1)^{n+1} \begin{vmatrix} \frac{B_2}{2!} 2^2 + \frac{1}{2} & 1 & 0 & 0 \cdot \cdot 0 \\ 2^4 \frac{B_4}{4!} & \frac{B_2}{2!} 2^2 + \frac{1}{4} & 1 & 0 \cdot \cdot 0 \\ 2^6 \frac{B_6}{6!} & 2^4 \frac{B_4}{4!} & \frac{B_2}{2!} 2^2 + \frac{1}{6} & 1 \cdot \cdot 0 \\ \cdot & \cdot & & \cdot \cdot \cdot \cdot \\ 2^{2n} \frac{B_{2n}}{(2n)!} & 2^{2n-2} \frac{B_{2n-2}}{(2n-2)!} & & \frac{B_2}{2!} 2^2 + \frac{1}{2n} \end{vmatrix}$$

Similarly, by the elimination of all the Bernoullian numbers except B_{2n} we get

$$2^{2n} \frac{B_{2n}}{(2n)!} = (-1)^{n+1} \begin{vmatrix} \frac{E_4}{3!} + \frac{E_2}{2!} & 1 & 0 & \cdot & 0 \\ \frac{E_6}{5!} + \frac{E_4}{4!} & \frac{E_4}{3!} & 1 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{E_{2n+2}}{(2n+1)!} + \frac{E_{2n}}{(2n)!} & \frac{E_{2n}}{(2n-1)!} & \cdot & \cdot & \frac{E_4}{3!} \end{vmatrix}$$

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Analytical proof of Gaskin's Theorem

THEOREM.—The circumcircle of a triangle ABC, self conjugate with respect to a conic cuts the director circle orthogonally.

1. Let the conic be $ax^2+by^2=1$ and let the vertices be $A(x_1\ y_1)$, $B(x_2\ y_2)$, $C(x_3\ y_3)$. Then we have

$$ax_1x_2+by_1y_2=1\,;\quad ax_2x_3+by_1y_3=1\,;\quad ax_3x_1+by_3y_1=1\,;\qquad \dots\ (1)$$

The equation of the circumcircle is

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{vmatrix}=0 \qquad \text{i.e. } x^2+y^2+2gx+2fy+c=0.$$

This cuts the director circle $x^2+y^2=\dfrac{1}{a}+\dfrac{1}{b}$ orthogonally if

$$g^2+f^2=g^2+f^2-c+\frac{1}{a}+\frac{1}{b} \qquad \text{i.e. if } c=\frac{1}{a}+\frac{1}{b}.$$

But $c=-\begin{vmatrix} x_1^2+y_1^2 & x_1 & y_1 \\ x_2^2+y_2^2 & x_2 & y_2 \\ x_3^2+y_3^2 & x_3 & y_3 \end{vmatrix} \div \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = -\dfrac{\Delta}{D}$

We shall show that the right hand side is equal to $1/a+1/b$.

$$D_1=\begin{vmatrix} x_1^2 & x_1 & y_1 \\ x_2^2 & x_2 & y_2 \\ x_3^2 & x_3 & y_3 \end{vmatrix}=x_1^2\,(x_2\,y_3-x_3\,y_2)+\ \cdot\cdot\cdot\ +\ \cdot\cdot\cdot \qquad \dots\ (2)$$

From the first and last of equations (1)

$$\frac{ax_1}{(y_3-y_2)}=\frac{by_1}{(x_2-x_3)}=\frac{1}{(x_2\,y_3-x_3\,y_2)}, \qquad \dots\ (3)$$

whence from the first and last ratios,

$$x_1^2\,(x_2\,y_3-x_3\,y_2)=\frac{1}{a}\,x_1\,(y_3-y_2).\ \text{etc}$$

$$\therefore\ D_1=-\frac{1}{a}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Similarly, we get

$$D_2=\begin{vmatrix} y_1^2 & x_1 & y_1 \\ y_2^2 & x_2 & y_2 \\ y_3^2 & x_3 & y_3 \end{vmatrix}=-\frac{1}{b}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Thus $c=1/a+1/b$.

2. If the conic is a parabola we take its equation in the form $y^2=4ax$ and in the place of (1) we get,

$$y_1 y_2 - 2a(x_1+x_2)=0; \quad y_2 y_3 - 2a(x_2+x_3)=0; \quad y_3 y_1 - 2a(x_3+x_1)=0 \ldots \quad (8)$$

We shall now have to show that

$$-\begin{vmatrix} x_1^2+y_1^2 & y_1 & 1 \\ x_2^2+y_2^2 & y_2 & 1 \\ x_3^2+y_3^2 & y_3 & 1 \end{vmatrix} \div \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 2a$$

We now get $D_1=0$, proceeding as before and D_2 turns out to be

$$-2a\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

3. The relation, $\dfrac{\Delta}{D} = -\left(\dfrac{1}{a}+\dfrac{1}{b}\right)$ yields the following geometrical interpretations:

$$OA^2\cdot\Delta BOC + OB^2\cdot\Delta COA + OC^2\cdot\Delta AOB = R^2\cdot\Delta ABC$$

where O is the centre of the conic and R is the radius of the Director Circle.

4. The problem may be treated purely as one in algebra: Given the relations (1), (2), (3) to show that the ratio of the determinants Δ and D is $-\left(\dfrac{1}{a}+\dfrac{1}{b}\right)$.

Both the theorem and the proof can be extended to space of higher dimensions.

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Do You Know

THAT the honey bee solved a problem in Calculus long before man could do it with all his Mathematics? The cell of a bee is hexagonal prism bounded at the inner end by three plane faces, each of which is a rhombus with angles 70° 32′ and 109° 28′. At the beginning of the 18th century, a mathematician Koenig determined the shape of the rhombuses so that the area should be least for a given volume of the bee cell. He found by Calculus that the angles were 70° 34′ and 109° 26′. Later investigations showed that Koenig had made a mistake in his calculation, but not the bees! The most economical angles are really 70° 32′ and 109° 28′.

From *the Annamalainagar Mathematics Exposition.*

Lecture Notes

1. *Integer roots by inspection.*

It is easily verified that the 5th powers of all the numbers 0, 1, 2,......9 and therefore also of all numbers, end in the same digits as the numbers themselves.

That is	$x^5 - x$	is divisible by 10	
Hence	$x^9 - x^5$	,,	,,
	$x^{14} - x^9$	,,	,,
	...	...	...

Therefore by addition, we get that x^{4m+1} ends in the same digit as x.

This result may be used to guess the $(4m+1)$th roots of numbers where the roots are integers of 1 or 2 digits.

Examples I. Find the 5th root of 9765625

The answer is 25, since the last digit must be 5 and the number lies between 20^5 and 30^5 (on cutting out the last 5 digits the number lies between $2^5=32$ and $3^5=243$)

The higher the root, the more easy it is to use the method. This can be extended to other roots, by using the extension of the above result namely that the $(4m+p)$th root of a number ends in the same digit as the pth root.

Examples II. Find the 7th root of 89741,0677851.

The 7th root ends in the same digit as the cube root, and the only number whose cube root ends in 1 is 1. ∴ the last digit is 1 and cutting off the last 7 digits, the remaining number lies between 5^7 and 6^7. So the answer is 51.

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2. *Lightning Elimination.*

I. If $x=by+cz,\ y=cz+ax,\ z=ax+by$ prove that

$$\frac{a}{1+a}+\frac{b}{1+b}+\frac{c}{1+c}=1.$$

Adding the three given equations

$$2\,(ax+by+cz)=x+y+z. \qquad \ldots (1)$$

By subtraction,

$$x\ (1+a)=y\ (1+b)=z\ (1+c)=k\ \text{(say)} \qquad \dots \quad (2)$$

Now

$$\frac{a}{1+a}+\frac{b}{1+b}+\frac{c}{1+c}=\frac{ax}{(1+a)\,x}+\frac{by}{(1+b)\,y}+\frac{cz}{(1+c)\,z}$$

$$=\frac{1}{k}(ax+by+cz)=1,$$

since adding of the three equations in (2) gives

$$x+y+z+ax+by+cz=3k,$$

which by (1) leads to $ax+by+cz=k$.

II. If $ax^2+bx+c=0$, $bx^2+cx+a=0$, prove that either

$$a=0, \text{ or } a^3+b^3+c^3=3abc.$$

Multiplying the first equation by x and subtracting the second, we get $a\ (x^3-1)=0$.

∴ either $a=0$ or $x=1$ or ω a cube root of unity.

Substitution in the given equations leads to $a+b+c-0$ or $a\omega^3+b\omega+c=0$ which in either case lead to

$$a^3+b^3+c^3-3abc=0.$$

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3. *On the formula* $\sum PA^2=\sum GA^2+3\ PG^2$.

Let $A_1\ B_1\ C_1$ be the mid-points of the sides of ABC, $A_2B_2C_2$ the mid-points of the sides of $A_1\ B_1\ C_1$ and so on. Then, from Apollonius Theorem we have $PB^2+PC^2=2PA_1^2+a^2/2$.

$$\therefore \qquad \sum PA^2=\sum PA_1^2+\tfrac{1}{4}\sum a^2$$

Similarly

$$\sum PA_1^2=\sum PA_2^2+\tfrac{1}{4}\sum a_1^2 \text{ where } a_1=B_1C_1 \text{ etc.}$$

$$\sum PA_2^2=\sum PA_3^2+\tfrac{1}{4}\sum a_2^2 \text{ where } a_2, b_2, c_2 \text{ are the sides of } A_2B_2C_2$$

Since G is the centroid of each of the triangles $A_nB_nC_n$ and is within the triangle, which is similar to ABC but of $1/2^n$ its size, $\sum PA_n^2 \rightarrow 3PG^2$ when $n\rightarrow\infty$.

Hence

$$\sum PA^2 = \frac{1}{4}\sum a^2 + \frac{1}{16}\sum a^2 + \frac{1}{64}\sum a^2 + \cdots \text{ to } \infty + 3PG^2$$

$$= \tfrac{1}{3}\sum a^2 + 3PG^2$$

But we know that

$$3\sum a^2 = 4\ (AA_1^2 + BB_1^2 + CC_1^2) = 9\sum GA^2$$

$$\therefore \quad \sum PA^2 = \sum GA^2 + 3PG^2$$

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4. *Sequences whose k^{th} differences form a geometrical progression.*

Let the kth order differences of a sequence u_n be the geometric progression a, ar, ar^2, $\cdots$ Then the succeeding order of differences also form geometrical progressions, whose first terms are

$$a(r-1),\ a(r-1)^2,\ a(r-1)^3\ \cdots\cdots$$

Hence $u_{n+1} = u_1 + {}_nC_1\ \Delta u_1 + {}_nC_2\ \Delta^2 u_1 \cdots + {}_nC_n\ \Delta^n u_1$

$$= u_1 + {}_nC_1\ \Delta u_1 \cdots + {}_nC_{k-1}\ \Delta^{k-1} u_1 + R_k$$

where $R_k = a\{{}_nC_k + {}_nC_{k+1}\ (r-1) \cdots + {}_nC_n (r-1)^{n-k}\}$

$$= \frac{a}{(r-1)^k}\left\{(1+r-1)^n - 1 - {}_nC_1(r-1) \cdots\cdots - {}_nC_{k-1}(r-1)^{k-1}\right\}$$

$$\therefore \quad u_{n+1} = \frac{ar^n}{(r-1)^k} + \left\{u_1 - \frac{a}{(r-1)^k}\right\} + {}_nC_1\left\{\Delta u_1 - \frac{a}{(r-1)^{k-1}}\right\} + \cdots$$

$$\cdots {}_nC_{k-1}\left\{\Delta^{k-1} u_1 - \frac{a}{(r-1)}\right\}$$

This method avoids the setting up of undetermined coefficients as in Hall and Knight's *Higher Algebra* p. 330 or working back from the order in G.P. as in Smith's *Algebra* p. 416.

Example: Find the nth term of the series

	u_n:	1	2	4	10	29	88	$\cdots$
Here	Δu_n		1	2	6	19	59	$\cdots$
	$\Delta^2 u_n$			1	4	13	40	$\cdots$
	$\Delta^3 u_n$				3	9	27	$\cdots$

Thus $\Delta^3 u_n$ is a G.P. and $\Delta^4 u_1$, $\Delta^5 u_1 \cdots$ are respectively 3, 3·2, 3·2², $\cdots$

$$\therefore \quad u_{n+1} = \frac{3^{n+1}}{2^3} + \left[1 - \frac{3}{2^3}\right] + {}_nC_1\left[1 - \frac{3}{2^2}\right] + {}_nC_2\left[1 - \frac{3}{2}\right]$$

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5. *The Nine points circle in homogeneous coordinates.*

1. The usual method of deriving the equation of the nine point circle is to take it as the circle passing through the middle points of the sides and the feet of the perpendiculars. Loney derives its equation by obtaining the lengths of tangents from the vertices to the Nine points circle (P. 72. Vol 2 *Trilinear Coordinates*). Askwith takes the general equation and writes down the condition for a circle and uses the definition of the Nine point circle. (*Analytical Geometry* P. 302-303). Baker uses the definition that the N. P. circle is a conic passing through the middle points of sides and through the circular points (*Principles of Geometry* 112-113. Vol 2). Here I derive its equation by equating the area of the pedal triangle of any point on the circumcircle to zero.

2. Let ABC be the given triangle H the orthocentre and DEF the pedal triangle of H. H has coordinates (Sec A, Sec B, Sec C) in Trilinears. Hence EF has Equation

$$y \cos B + z \cos C - x \cos A = 0$$

and so on for FD and DE. Let P (α, β, γ) be a point on the circumcircle of DEF and let PL, PM, PN be the perpendiculars from P to EF, FD, DE respectively.

Then area of triangle MPN varies as

$$\tfrac{1}{2} \sin 2A\,(\alpha \cos A + \gamma \cos C - \beta \cos B)(\alpha \cos A + \beta \cos \beta - \gamma \cos C) \div G$$

where $G = \cos^2 A + \cos^2 B + \cos^2 C + 2 \cos A \cos B \cos C$,

so that (α, β, γ) lies on the locus:

$$\sum (\alpha \cos A + \gamma \cos C - \beta \cos B)(\alpha \cos A + \beta \cos B - \gamma \cos C) \sin 2A = 0.$$

i.e. $$\sum \alpha^2 \cos^2 A\,(\sin 2A - \sin 2B - \sin 2C) + 4 \cos A \cos B \cos C \sum \beta\gamma \sin A = 0.$$

i.e. $$\sum a \cos A \cdot \alpha^2 - \sum a\beta\gamma = 0.$$

(i) This incidentally shows that the Nine point circle passes through the intersections of circumcircle and polarcircle, because $\sum a\beta\gamma = 0$ is the circumcircle and $\sum a \cos A\ \alpha^2 = 0$ is the polarcircle.

(ii) That the Radical axis which is the axis of perspective is

$$\alpha \cos A + \beta \cos B + \gamma \cos C = 0.$$

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REVIEWS

The non-singular cubic surfaces

BY B. SEGRE, published by Sir Humphrey Milford at the Oxford University Press, 1942, Pp 180, Price 15/- net.

Fashions change in Mathematics as in other objects of human interest and the complicated and many sided symmetries connected with the 27 lines on a cubic surface (first discovered by Cayley and Salmon in 1849) and which were keen objects of study by mathematicians a hundred years ago, have given way to others of a more abstract character in which the concept of "group" and of "continuity" have played an increasingly dominant role culminating in the creation of that most fruitful and unclassifiable subject of eager study at the present day, topology. In this work the talented author brings to the study of the cubic surface a new method based on the principle of continuity and of the limiting forms of the 27 lines when the surface degenerates into three planes. leading to a very convenient notation and graphical representation of the lines. With great skill and clarity he depicts the configuration of skew lines, double sixes, tritangent planes etc. connected with the cubic and proceeds to the study of the group of order 51840 of the 27 lines. The best part of the work is Chapter III where considerations of continuity are used to classify real non-singular cubic surfaces into five connected continuous systems F_i and of all real double-sixes into four continuous systems. It is shown that while four of the systems form closed connected manifolds (in projective space) the fifth F_5 consists of two pieces of which one is ovaloidal. The 27 lines decompose the cubic surface into cells, and the topology of the resulting generalised polyhedron is studied. It is interesting to note that one of the types (F_4) and the non-ovaloidal part of F_5 are homeomorphic with the projective plane.

The last chapter is predominantly analytical and deals with the reduction of the general non singular cubic to a canonical form, with the classification in the complex as well as the real field of cubics which possess auto-projective transformations and allied topics.

An exhaustive monograph by a master Geometer from which one may learn the kind of problems suggested by the development of modern Goemetry on studies cnnsidered to have been completed in the past, and the skill and the patience with which an expert geometer will handle them.

A. N. RAO.

Lectures on the Theory of Functions

BY J. E. LITTLEWOOD, published by Sir Humphrey Milford at the Oxford University Press, 1944; Price 17/6 pp. 243.

Books in English dealing with advanced topics in the theory of Analytic Functions are not many. The book under review is a welcome addition to this small number.

The book consists of an introduction of more than one hundred pages and two chapters.

Every one who aspires to do research work in Higher Analysis must be thoroughly familiar with almost all the results discussed in the introduction. This is probably the first time when these results are found collected together in a compact form in a single book these being usually found scattered in various treaties or journals. The topics discussed range over a wide field including Inequalities, standard results on Lebesgue Integrals, Fourier Series and Harmonic Functions.

Chapter I deals with the classical results in the theory of Analytic Functions beginning with the maximum modulus principle and leading to the existence theorem of the theory of Conformal Representation and some results on the boundary problem in simple cases.

Chapter II contains a good deal of the author's own valuable contributions to the problem of missing values. The general problem is to discuss the behaviour of functions regular in a circle and omitting to take entirely or more than a given number of times certain set of values. The problem is attacked by using the general principle of Subordination which is also useful in other problems, for instance, in the theory of Schlicht functions. The printing is good and the get up is attractive.

V. G. IYER

Jacobian Elliptic Functions

BY Eric Harold NEVILLE, published by Sir Humphrey Milford at the Oxford University Press, 1944. pp. XIII+331 Price 25/-

The Jacobian Elliptic functions sn u, cn u and dn u were first studied in connection with the inversion of elliptic integrals. But in modern times, the tendency has been to introduce them by means of Theta Functions and use the methods of function theory to study their properties.

The book under review emphasises this function theoretic approach to the Jacobian Elliptic Functions while, at the same time, it devotes full attention to the inversion problem. The double periodicity of the Jacobian Functions is the starting point of the method of treatment adopted in the book.

The introductory chapter lays the foundation for the results of later chapters by studying lattices and the doubly periodic (elliptic) functions associated with these lattices. The classical Weierstrassian function with a double pole at the vertices of a lattice is studied in detail and every elliptic function is expressed in terms of this function.

Chapters I to IV are a natural sequence to the introduction. The author constructs and studies the properties of the twelve elementary functions with two simple poles instead of a double pole in a period parallelogram. Though these functions are not new, their systematic study and use in connection with the classical Jacobian functions is an important and novel feature of the book under review.

Chapter V to VIII are devoted to a thorough investigation of the problem of inversion of elliptic integrals. The difficulty inherent in the problem is clearly set forth, each point is separately discussed and the final solution is completed in the course of these four chapters. chapter IX deals with functions and integrals with real critical values.

Chapters X to XV deal with the subject proper of the book, namely Jacobian Elliptic Functions. These functions are obtained from the elementary functions discussed in chapters I to IV by specialisation of the lattice. The addition theorems and other properties follow naturally from those for the elementary functions. The Jacobi and Landen transformations are studied in chapter XIII. The next chapter deals with the integration of the Jacobian Functions and chapter XV with the dependence of these functions on the parameter.

Chapter XVI discusses Theta Functions and their relation to Jacobian Functions and the last chapter is devoted to real functions and integrals. The book concludes with a good number of exercises and notes on these exercises.

Another noteworthy feature of the book is the use of new and suggestive and very helpful notations for the various functions and parameters involved.

The printing and get up is excellent. The book is to be welcomed as the first standard treatise on Jacobian Elliptic Functions approached from the function-theoretic point of view.

V. G. IYER.

Algebra for Intermediate

By S. K. ABHYANKER, Professor, Victoria College, Gwalior
The Indian Press, Allahabad 1943, Pp. 285.

Dynamics for B. A., B. Sc. & F. E. Classes

By S. NARAYANAN, Lecturer Loyola College, Madras.
Rockhouse & Sons, Madras, 1943 Pp. 226.

The impact of war conditions on India, as a result of which it is difficult to get even well known text books from abroad, has accelerated the production of text books by Indian authors,' though at present it is confined mostly to those for the Intermediate and B. A. pass classes. It is gratifying that of late, teachers of experience like Prof. Abhyanker and Prof. S. Narayanan are coming out with books which integrate their long teaching experience and seek all possible avenues for improving the presentation of the usual subject matter which one expects in such books.

Prof. Abhyanker's book opens with a revision chapter devoted to real numbers, their evolution and properties. Then follow the usual chapters on indices, the development of the function concept with a graphical study of simple functional relations, quadratics etc. Complex numbers are defined as real number pairs, and the book ends with limiting values and explanation of how logarithms may be calculated. A welcome feature of the book is the occasional historical footnote, and the reference to Sreedharacharya's method of solving a quadratic by reducing it to the form $(2ax+b)^2=b^2-4ac$ with the appropriate quotation. One feels that text books generally, will gain in human interest by incorporation of material connected with the history of the subject especially where the methods explained are related to the work of ancient Indian mathematicians. The printing and get up are good.

Mr. Narayanan's book on Dynamics follows the conventional lines of Loney's Elementary Dynamics, but uses calculus methods which are now part of the normal equipment of the student who has passed the Intermediate examination of most Indian Universities. It contains chapters devoted to laws of motion, projectiles, impact, plane motion and of a rigid body rotating about an axis.

In defining the fundamental concepts of dynamics, critical writers replace the Newtonian definition of mass as the "amount of matter" in a body, and prefer to define mass ratio through the third law of motion as the inverse ratio of their mutually induced accelerations. The author defines mass by $m=w/g$, where w is measured by a spring balance. This has the advantage of being more in accord with the actual method of comparing masses commonly in use, though it blunts the concept of mass as applied to celestial bodies.

Among the many problems worked out in the book the reviewer finds the following, taken from a Madras University Examination paper:

"Two equal billiard balls are in contact on a table; a third equal ball strikes them symmetrically and remains at rest after the impact. Show that the coefficient of restitution is 2/3." The author gets the answer, as the examiner expects, by using the principle of conservation of momentum for all the three balls, and the law relating to the relative velocity along the common normal to two of the balls. But is the application of the latter law (which is enunciated only for two bodies) to the case of simultaneous impact of three bodies justifiable? As intervals below a certain critical interval are indistinguishable from simultaneity, one feels that problems of simultaneous impact should be derived by a limiting process from two distinct impacts separated by an infinitesimal interval. That the answer would be incorrect if viewed thus is seen from the fact that the first ball cannot come to rest after its last impact unless $e=1$.

The book is well written and deals with the topics it contains in an able and lucid manner, and should prove very helpful to those for whom it is intended.

A. N. RAO.

Elementary Mathematical Astronomy,

BY BARLOW AND BRYAN, revised BY SIR HAROLD SPENCER JONES Astronomer Royal, University Tutorial Press, London, 1944, Pp. 387.

This is the fifth edition of a book born half a century ago and still going strong as a standard text book, completely revised and brought up to date by the Astronomer Royal. There has been a rearrangement of chapters bringing together those on Refraction, parallax and aberration, while a chapter has been introduced on precession and nutation.

The precision of modern time keepers has made it necessary to introduce a distinction between "apparent sidereal time" and "Mean sidereal time". This distinction is similar to that between apparent time and mean time and is due to a similar reason, the non-uniform motion of the first point of Aries on the equator owing to the combined effects of precession and nutation. We imagine a "mean equinox" travelling uniformly along the equator and smoothing out the irregularities of the true equinox and never far from it. The extreme difference in R. A. (termed the "nutation in right ascension and corresponding to the "equation of time") turns out to be ± 1.2 seconds of time. A good precision clock used in modern observatories will give a time more uniform than "apparent sidereal time"; hence the necessity for the distinction. Closely related to the above is the distinction between the true place of a star (its heliocentric position referred to the true equator and the true equinox at the moment of observation), and its mean place (heliocentric position referred to the mean equator and equinox at the beginning of the year). The opportunity of revision has also been utilized to change the sign of the "equation of time" so as to accord with the present day Nautical Almanac.

One of the primary urges towards the study of astronomy was navigation, and today when aerial navigation has become so common, the necessary data for drawing the position circle of an aviator is given in the *Air Almanac* a description of which is included in this edition, as also that of "the bubble sextant" and refinements of methods for finding the position of an observer whether on sea or in air. Owing to the absence of a well defined horizon and the large correction for dip from a height not known with sufficient accuracy, aviators use the "bubble sextant" in which a bubble of air in a spherical level filled with liquid determines the vertical direction as the radius through the bubble (the aircraft maintaining a steady speed in a straight line). By reflection in a mirror inclined at an angle of 45°, the rays of light from the bubble are rendered horizontal and thus provide the artifical horizon for determining the altitudes of celestial observed bodies by means of the sextant.

We wish this rejuvenated edition of Barlow and Bryan's book an Astronomy another long lease of usefulness which it eminently deserves.

A. N. R.

GEORGE DAVID BIRKHOFF, 1884—1944.

By last airmail came unexpected news that the leading American mathematician Prof. G. D. Birkhoff [not to be confused with his distinguished son and colleague Garrett Birkhoff] of Harvard had suddenly passed away. It is unfortunate that this note, which could have served as tardy appreciation on the occasion of his sixtieth birthday (March 21 1944) should have to be turned into a rather slight obituary.

Birkhoff's principal significance in contemporary science was that with him American mathematics came of age. He first reversed the general trend of a pilgrimage to Europe for deeper mathematical studies which was considered essential for every serious American student before him. When continental mathematicians first referred to his work, it was (as for example Wilhelm Blaschke) with the emphatic qualification "der Amerikaner Birkhoff" and that citation, I feel certain, must have been regarded as a greater triumph by Birkhoff himself than the innumerable honorary degrees, fellowships of academies and learned societies (led by the Academie des Sciences at Paris), prizes (none of greater distinction than that of the newly opened pontifical academy) showered upon him throughout his abruptly terminated scientific life. Not that he was not an internationalist, for he did his very best to save Göttingen as a mathematical centre during the depression, before Hitler destroyed all such hopes. After that he was instrumental in bringing to the USA the finest of the exiled talent, though he could never have been mistaken for a pro-semite by anyone who spoke with him for more than thirty seconds on the subject. Nevertheless, he was and always remained an American, at times aggressively so, with forthright direct action and oversimplified thinking-except in his mathematics. When Indians like T. Vijayaraghavan and S. Chandrasekhar received the courtesy that he always lavished upon all whom he believed to possess any degree of scientific competence, Birkhoff would manage to persuade himself that he was doing it for the good of America and not because of his own genuinely kind and hospitable nature.

To appreciate what this means, without letting it go as a grudging compliment to the memory of a great scholar, one would have to study the mathematical atmosphere of Birkhoff's younger days when the very claim to be an American scholar exposed the claimant to ridicule at home and polite contempt abroad. The true scholars of his formative period were either imported, like Sylvester and Bolza, or were isolated and frozen out for lack of appreciation. Josiah Willard Gibbs had to be resurrected by the German chemist Ostwald before the USA realised that it had possessed a great scientist. Eliakim Hastings Moore, Birkhoff's beloved and most influential teacher, found his "general analysis" totally neglected till the fame of his distinguished pupil directed the attention of others there-to. By that time it was much too late for the method to be effective or Moore's notation to be

adopted, for Hilbert's followers had skimmed all the cream off that particular jug. This was inevitable in the undeveloped state of American mathematics, though Moore himself was the first in any country to realize the full connotation of Hilbert's work on integral equations which has had so profound an influence upon current scientific activity through modern quantum theory. Birkhoff studied under Moore at Chicago in 1902, came to Harvard (the only other American mathematical centre then existing) next year with Moore's approval, worked rather unenthusiastically for two years, and returned to Chicago for another two years of real inspiration. The stage at Harward was then fully occupied by the imposing presence of B.O. Peirce who made up for a total lack of mathematical intelligence by his preternatural solemnity and an impressive beard [which earned him the soubriquet of "Santa Claus" from Birkhoff's generation of students]. Peirce's ostentation overshadowed the real mathematical ability of his son Charles, whose unconventional life and thought (like that of G.W. Hill) led to complete intellectual obscurity and a miserable academic career with the doubtful compensation of posthumous honour. Far better as a teacher was William Fogg Osgood, a much younger member of the Harward mathematical department which he succeeded in raising to pre-eminence in latter days. Osgood had an excellent mathematical training under Felix Klein, was instrumental in bringing the modern analysis of Weierstrass with its battery of epsilons and deltas to bear upon the American student, knew what the Erlanger programme was, possessed the adjuncts considered necessary for a good teacher in his day including beard, pompous mannerisms (plus the affectionate student-bestowed tittle of "Foggy Bill") and lack of imagination. He failed to inspire Birkhoff, though he did succeed in bringing him as a professor to Harward from Princeton in later days, and Birkhoff treated him with all the formal respect due to a teacher. But their mutual regard went no further. Osgood privately characterized Birkhoff's presentation as "sloppy", which it certainly was, and hence by innuendo his thinking too, which was certainly never sloppy. It is to be noted that these two leaders of scientific endeavour managed to keep their personal views about each other from interfering with their work and close cooperation in the same University department; this is a lesson that has yet to be learned by academicians in our own country. The last of Birkhoff's teachers worth mentioning was Maxime Bôcher whose crystal clear lectures derived from his crystal clear thinking. The very perfection of these lectures, however, left the budding young research worker cold, with unfortunate after effects; for, Birkhoff's own lectures left very much to be desired even twentyfive years later. Those of his lectures that the advanced students did understand were invariably considered by them the most inspiring that they had ever heard, but usually he lost himself, the subject, and the audience in his own latest brainwave which might have developed that very morning between the breakfast table and the lecture room.

The rest of Birkhoff's story is precisely that of the growth of modern mathematics in the USA, of the substitution of a few isolated individual workers by the general development of indigenous talent and a powerful *school*

of mathematics which rapidly took the lead over all others soon after the first world war. The best view of this development as regards persons and performance is to be had from Birkhoff's own semicentennial address [1938] to the American Mathematical Society, "Fifty Years of American Mathematics". In this his thorough grasp of many branches of the subject shows itself, along with his breadth of vision, though he consistently minimizes his own contributions and the reader will not realize that a goodly number of those he praises with justice were made by his own brilliant students, often with his direct inspiration. He was early associated with Vandiver in work on Fermat's last theorem, and his pleasure was unbounded when he saw in the advance copy of Landau's book on number theory that a whole section had been devoted to the work of his early collaborator, gaining him long-delayed recognition. What first made Birkhoff himself internationally famous was his simple proof of Poincaré's geometric theorem on the existence of invariant points for a ring transformed into itself with the two boundaries advanced in opposite direction. Poincaré died without proving the theorem and a proof supposed to have been discovered by Phragmén actually came to nothing, leaving Birkhoff in sole possession of the field and the eminence that was his due. Before this, his work on existence theorems for difference equations, showing their close parallel behaviour with differential equations had given him substantial reputation. He followed this success up with further important work on dynamics, a continuation of Poincaré's ideas which approached the dynamical problems of existence of stable motions, of periodic orbits, etc. by methods of topology and differential geometry *im Grossen*. This was, perhaps, his most important and successful work. His attack on the four colour map problem was not successful in spite of intensive study and effort, but it inspired others like P. Franklin to make their own important advances. In mathematical physics his book on relativity and modern physics deserves to be far better known than it is, though it came rather late in the relativistic day, his contributions there have been regarded as important by no less a geometer than Levi-Civita. If he did not follow up the enticing field of tensor analysis, it must undoubtedly have been because the Princeton school (which he himself helped found with Veblen) had plunged into the heart of the subject with greater vigour and success than could have been expected at that time by a solitary research worker whose main interest lay in other branches of mathematics. Birkhoff's proof in 1931 of the ergodic theorem was due to his having gone first to the main point of the topic towards which such capable workers as J. von Neumann were also converging. The actual flawless proof was supplied a year later by the Soviet mathematician Khinchin in a definitive paper which is at the same time a model of graceful tact; but the basic ideas derived admittedly from Birkhoff who showed the fundamental position of the concept of measure and integration, particularly of Lebesgue integration, in such problems. On the other hand, I find it impossible to take Birkhoff's "Aesthetic Measure" seriously.

Authoritative and complete obituaries will be written by those who knew Birkhoff much better than the present writer, and whose mathematical stature

could be said to equal his in some directions at least. One can only point to Birkhoff here as an important subject for study in the development of science in a country which changes over rapidly to an advanced mode and greater concentration of production. The precise function of the individual can only be studied against the background of the changing system, and this is generally not done in obituaries of the type to which we are accustomed. India must some day take the lead in a similar way—though a tremendous expansion of business is now going on in this land without the corresponding scientific advance one should naturally expect, for the simple reason that our intellectual life is still disoriented and based (even debased!) upon concepts originally derived from the scanty overflow of a colonial economy. If an answer is wanted some day as to why a real school of Indian mathematicians (did or) did not develop after the second world war, the historian of science could do worse than study the development of Birkhoff as a facet of the development of his country. He used to mention in moments of confidence (expressing amusement mingled with slight but noticeable regret) that a Dutch relative of his had offered him an active share in a cable company, and had the laugh of him ever since by making a fortune out of the enterprise to which Birkhoff had preferred the obscurity of a simple professorship! One wonders what had happened to the Dutch cousin and the cable business by the end of 1940. But that must have been a year of disappointment for Birkhoff too, for the international mathematical congress scheduled for Harvard that year had to be cancelled because of the war. It would have been an occasion to round out his whole career, an occasion upon which he could see his students carry his country to the forefront of the science to which he had devoted his entire life, giving up all the financial opportunities that were open to any American of his mental powers in the period of expanding bourgeois capitalist economy. One wonders whether the war had changed Birkhoff's naive views about the importance of race and the essential glory of Nordics, which he held very strongly at least till 1934. Did he realize that "pure Anglo-Saxon" was even more meaningless than "American scholar" had been before him, and that "American" was not a race but a mentality? It shows the fundamental nobility of his character that he never allowed such views to interfere with his scientific judgment, nor to prejudice him in the slightest in the matter of adjudicating research fellowships and prizes. He had no hesitation in recommending for important posts people with whom he was not and did not want to be on visiting terms. It was noticeable that when he expressed his views on politics or sociology in any sort of mixed company, anyone could (and someone often did) contradict him flatly without offence on either side. But the moment he began to talk about mathematics, the others (no matter what their speciality) quietly stopped their own chatter to listen. It was impossible not to love such a teacher.

In the lifetime of any scientist who has not the good fortune to die young, (like Abel, Galois, Riemann, Ramanujan) there must inevitably come a moment

when he begins to sense his own limitations, to feel the joints of his mind stiffening no less than those of his body, to shrink a little from the thought of testing his ability against totally new problems. Far too many scientists of first rank then go the way of popular acclaim, newspaper reputations and public performances, to drown the insistent voice from within which begins to ask "the end is near, you are going down the intellectual slope; how much of your work will survive? What is your share of 'immortality?". If Birkhoff had any inkling of the end, one feels that he would have faced it without a qualm, for his work was safe (in spite of wartime regression) in the hands of the American school. His great contemporaries probably greater in absolutely effective total scientific achievement as for example Hilbert, Levi-Civita, Lebesgue, Borel, could not have had this assurance of continuity dying as they did in the midst of universal distress and the collapse of all they had been brought up to honour. In this, Birkhoff was fortunate above the rest, and none deserved such fortune better.

Poona D. D. KOSAMBI

GLEANINGS

Visual Geometry

Have you noticed how when you look out through a railway window you see objects flowing fast by? While they move, they seem to change their shape and size, becoming bigger as they come near. The motion seems to consist of a gradual increase in size (dilatation), a linear backward velocity (translation) and a twisting and turning (rotation.)

You can see here objectively on this glass flask how the shape and size of objects appears gradually to change.

Piloting in the Air

There are four main methods for piloting an airship.

(i) observing landmarks and recognising them by maps;

(ii) directing the plane towards destination and proceeding till the distance to the same is covered;

(iii) understanding the directions of the main broadcasting stations sending radio beacons and thereby fixing one's position;

and (iv) fixing the latitude and longitude by celestial observations.

From *The Annamalainagar Math. Exposition.*

ANNOUNCEMENTS AND NEWS

The following gentlemen have been admitted as members of the Indian Mathematical Society :—

Dr. U. Sivaraman Nair, M.A., Ph. D., F. A. Sc., Head of the Statistics Department, Travancore University, Trivandrum,

S. M. Karmalkar Esq. M.A., Asst. Master, Government High School, Akola,

V. S. Huzurhazar Esq. M.A., Lecturer, Rajaram College, Kolhapur,

V. Narasimhamurti Esq. M.A., Lecturer, Andhra University, Guntur,

Dr. B. N. Prasad, Ph., D., D. Sc. Dept. of Mathematics, Allahabad University, Allahabad,

K. D. Bhattari Esq. M.A., of B 1/192, Dumaria, Kothi, Assi, Benares,

B. Suryanarayana Rao B. E. Section sub-officer, Yelahanka,

The Committee of the Indian Mathematical Society has been reconstituted for the years 1944—46 as follows:

Prof. F. W. Levi, *President*, Prof. A. Narasinga Rao, *Treasurer*, Prof. D. D. Kosambi, *Librarian*; Other members: Dr. R. Vaidyanathaswami, Madras, Dr. N. R. Sen, Calcutta, Dr. A. N. Singh, Lucknow, Prof. J. Maclean, Bombay, Dr. Ram Behari, Delhi, Dr. S. C. Dhar, Nagpur, Dr. S. S. Pillai, Calcutta, Prof. S. Mahadevan, Madras, and Dr. M. R. Siddiqi, Hyderabad.

Prof. S. Mahadevan and Prof. M. R. Siddiqi have been appointed Honorary Secretaries for the next two years. The committee of the Society has passed resolutions expressing its appreciation of the work of the retiring office bearers Prof. L. N. Subramaniam, retiring Treasurer and Prof. R. P. Shintre retiring librarian for their valuable services to the Society while in office.

Competitors for the "Narasinga Rao Medal for Mathematical research" are reminded that the latest date for the submission of solutions of the Prize Problem on Topology is the 31st of July 1945. Solutions should be sent to Dr. A. Narasinga Rao, Annamalai University, Annamalainagar. The conditions governing the competition are given on pages 61 and 62 of The Mathematics Student, March and June 1943.

THE INDIAN MATHEMATICAL SOCIETY: STATEMENT OF ACCOUNTS FOR 1942

RECEIPTS

	Rs.	As.	Ps.
Opening balance			
Fixed Deposit ...	10,000	0	0
Savings Bank (i) ...	2,510	6	4
,, (ii) ...	586	13	5
Current Acct. ...	238	2	11
Subscriptions			
Life members ...	185	0	0
Ordinary ...	630	0	0
Concessional ...	630	0	0
Journal and *Student* ...	646	3	0
Sale of back numbers etc. ...	189	12	6
Grants-in-aid			
Osmania University ...	100	0	0
Madras University ...	150	0	0
Bombay University ...	200	0	0
Interest receipts ...	525	15	8
Rs. ...	16,592	5	10

EXPENDITURE

	Rs.	As.	Ps.
Joint Editor ...	23	7	0
Hon. Librarian ...	200	0	0
Hon. Treasurer ...	18	13	0
Hon. Secretary General ...	110	0	0
Do. Journal ...	163	9	9
Printing of *The Journal* and *The Student* ...	1,413	9	3
Bank Charges ...	8	13	0
Closing balance			
Fixed Deposit ...	10,000	0	0
Savings Bank (i) ...	2,998	14	1
,, (ii) ...	1,295	15	4
Current Acct. ...	359	4	5
Rs. ...	16,592	5	10

L. N. SUBRAMANIAM

Hon. Treasurer

8-7-1944

G. A. SRINIVASAN

Checked and found correct

8-7-1944

THE INDIAN MATHEMATICAL SOCIETY: STATEMENT OF ACCOUNTS FOR 1943

RECEIPTS

	Rs.	As.	Ps.
Opening balance			
Fixed Deposit ...	10,000	0	0
Savings Bank (i) ...	2,998	14	1
,, (ii) ...	1,295	15	4
Current Acct. ...	359	4	5
Subscriptions			
Life members ...	75	0	0
Ordinary ...	390	0	0
Concessional ...	498	0	0
Journal and *Student* ...	595	11	0
Sale of back numbers etc. ...	276	13	0
Reprints ...	19	2	0
Miscellaneous ...	2	4	0
Grants-in-aid			
Osmania University ...	100	0	0
Madras University ...	150	0	0
Bombay University ...	200	0	0
Dr. R. Alagappa Chettiar ...	100	0	0
Interest receipts ...	537	3	5
Rs. ...	17,598	3	3

EXPENDITURE

	Rs.	As.	Ps.
Joint Editor ...	69	7	0
Hon. Librarian ...	200	0	0
Hon. Treasurer ...	10	0	0
Hon. Secretary General ...	133	13	6
Do. Journal ...	45	0	0
Brues and Bawes ...	57	6	0
Papèr bill ...	220	9	0
Printing of *The Journal* and *The Student* ...	990	1	0
Bank charges ...	8	6	0
Closing balance			
Fixed Deposit ...	10,000	0	0
Savings Bank (i) ...	3,781	2	0
,, (ii) ...	1,752	11	10
Current Account ...	329	10	11
Rs. ...	17,598	3	3

L. N. SÙBRAMANIAM
Hon. Treasurer
8-7-1944

G. A. SRINIVASAN
Checked and found correct
8-7-1944